Such relations are thus derived for certain symmetric six-rowed determinants. $D_6$, however, is of a highly specialized type; it is the discriminant of the sum of three 6-ary squares. Is the relation (2), established for this special type, valid for all symmetric determinants of six rows? It is; for it involves no constituent from the principal diagonal, so that the 18 parameters of our $6 \times 3$ array are available for representing the 15 constituents $\left(\frac{6 \cdot 5}{1 \cdot 2} = 15\right)$, lying outside the principal diagonal of any given symmetric 6-rowed determinant.

The same consideration can be relied upon in adapting this proof to any Kronecker relation among $m$-rowed minors of a $2m$-rowed symmetric determinant, since always

$$\frac{2m(2m-1)}{2} < 2m^2.$$

For convenience of reference, I subjoin the general form of this relation as enunciated by Kronecker, *loc. cit.:*

$$|a_{gh}| = \sum_r |a_{ik}|,$$

$$\left(\begin{array}{ll} g=1, 2, \cdots, m; & h=m+1, m+2, \cdots, 2m \\ i=1, 2, \cdots, m-1, r; & k=m+1, m+2, \cdots, r-1, m, r+1 \cdots 2m \end{array}\right).$$

NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS, *December* 10, 1895.

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# ON THE LISTS OF ALL THE SUBSTITUTION GROUPS THAT CAN BE FORMED WITH A GIVEN NUMBER OF ELEMENTS.

BY DR. G. A. MILLER.

T. P. KIRKMAN published in 1863 the first extensive list of all the transitive substitution groups that can be formed with a given number of letters. A number of interesting facts are associated with this list. Before entering upon its discussion we shall give a brief account of the more important direct steps towards the formation of such lists.

Paolo Ruffini published a work* in 1799 in which we do

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* The complete title of this work in two volumes is, "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto, di Paolo Ruffini,*" Bologna, 1799.

not only find groups divided into transitive,* intransitive, primitive and non-primitive, but also an attempt to give all the orders of the groups that can be formed with five elements. This work is especially interesting because it contains some concepts which have been attributed to Cauchy for a long period of years.†

In 1845 Augustin Cauchy published in *Comptes Rendus,* vol. 21, pp. 1363–1369, lists of all the possible orders of the groups that can be formed with two, three, four, five, and six letters. He determines the orders of the intransitive, non-primitive and primitive groups separately. His lists contain a few orders for which no group of the given class exists, and the list of the orders of the non-primitive groups of degree six does not contain 36. There are no other omissions. Some theorems relating to group construction precede these lists.

In 1850 J. A. Serret published in *Liouville's Journal* complete lists of all the substitution groups that can be formed with four and with five letters. No special notation is employed, but the groups are written in ordinary language in the summaries on pages 52, 53 and 65–67. The facts that these lists are complete and that they contain the groups themselves instead of their orders only are evidences of a significant step forward even if the author confined himself to a very small number of letters.

Emile Mathieu continued the lists which Cauchy began, by enumerating all the possible orders of groups which contain seven and eight letters. These lists are given in *Comptes Rendus,* vol. 46, pp. 1048 and 1208 respectively.

### *Kirkman's List.*

This is found in *"Proceedings of the Literary and Philosophical Society of Manchester"* (1863), vol. 3, pp. 133–152.‡ A table of corrections containing ten additional groups and some changes of the notation of the groups in the list is given by the same author on p. 172, vol. 4, of the same journal.

We reproduce below, the first part of this list, through degree

* Cf. JORDAN, *Traité des substitutions,* preface VII.; MATHIEU, *Comptes Rendus,* 46, pp. 1048 and 1049.

† Cf. H. BURKHARDT, *Schlömilch's Zeitschrift,* 1892, supplement, p. 159. In the same article is found a discussion of the part of Ruffini's work which relates to substitution groups, pp. 133–137. We refer to this article for a more complete account of Ruffini's work.

‡ The heading of the article in which this list is given is, "The complete theory of groups, being the solution of the mathematical prize question of the French Academy for 1860. By the Rev. Thos. P. Kirkman, M.A., F.R.S., and honorary member of the literary and philosophical societies of Manchester and Liverpool."

four, for the purpose of adding clearness to the following remarks. It may be observed that the group of two letters is not given in the list.

$$3\cdot 1=1+2_3,\quad Q=1;\quad 3\cdot 2=1+2_3+3_{21},\quad Q=1,$$
$$4\cdot 1=1+2_4+1_{2^2},\quad Q=3;\quad 4\cdot 2=1+2_4+3_{2^2}+2_{21^2},\quad Q=3,$$
$$4\cdot 3=1+8_{31}+3_{2^2},\quad Q=1;\quad 4\cdot 6=1+8_{31}+3_{2^2}+6_4+6_{21^2},\quad Q=1.*$$

*Explanations of the notation.* The first members of these equations give the orders of the groups. The second members give the enumeration of the different types of substitutions found in the groups. The subscripts indicate the degree of the cycle or cycles that compose the substitutions of the given type. $Q$ stands for the number of conjugates of the group in the corresponding symmetric group.

The list and the table of corrections contain all the transitive groups through degree seven with exception of the group of degree two mentioned above. The following six groups of degree eight are omitted.

| ORDER. | NUMBER. | GROUP. |
|---|---|---|
| 16 | 1 | $\left\{\begin{array}{l}(ae\,.\,bf\,.\,cg\,.\,dh)B'(abcd\,.\,efgh) \text{ or} \\ (aebf\,.\,cdgh) \text{ cyc } (ac\,.\,bd)(ef\,.\,gh),\end{array}\right\}$ |
| 24 | 1 | $(abcd\,.\,efgh)$ pos $(ae\,.\,bg\,.\,cf\,.\,dh)$, |
| 96 | 1 | $(ae\,.\,bf\,.\,cg\,.\,dh)\{(abcd)_{12}(efgh)_{12}\}$ ter, |
| | 2 | $(ae\,.\,bg\,.\,cf\,.\,dh)\{(abcd)_{12}(efgh)_{12}\}$ ter, |
| 192 | 1 | $\{(abcd)$ all $(efgh)$ all$\}$ sex $(ae\,.\,bf\,.\,cg\,.\,dh)$, |
| 576 | 1 | $\{(abcd)$ all $(efgh)$ all$\}$ pos $(aebf\,.\,cg\,.\,dh)$. |

The two given groups of order 16 are conformal † and they have the same number of conjugates in the symmetric group of degree eight. It is, therefore, impossible to determine which of these is omitted, since Kirkman's notation for the two is exactly the same. The fact that his notation involves such uncertainty is an objection to it. A more serious objection is furnished by the fact that the notation does not enable one to write directly the substitutions of the group.

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* The list does not contain the four-group. This is given in the table of corrections together with two missing groups of degree six, viz., $(abcdef)_{12_2}$ and $\pm(abcdef)_{24}$. The list is complete for the degrees three, five, and seven.

† For brevity we use the term "conformal groups" for non-conjugate groups, all of whose substitutions are of the same type. Kirkman does not give any conformal groups in the list itself. The table of corrections, however, contains a group which is conformal to one in the list, viz., a $G_{32}$. Cf. KIRKMAN, *Mathematics from Educational Times*, vol. 38, pp. 80, 81.

It appears quite likely that Kirkman regarded the two given groups of order 16 as conjugate without going through the full proof. Two other omissions may be accounted for in almost the same way. Kirkman's group of order 96, and the first group of this order given above, as well as the two positive transitive groups of order 192, are also conformal. In each of these cases the omitted group has 210 conjugates in the symmetric group of degree eight, while the given group has only 105, so that it is easy to determine which of the conformal groups are in the list.

The other three omitted groups cannot be accounted for in the same way, since Kirkman's notation for these groups would be quite distinct from that used for other groups in the list.

It may be observed that while Kirkman omitted six transitive groups of degree eight there are twenty-four such omissions in Cayley's list,* which was published about thirty years later.

The six groups, which are not given in Kirkman's list and table of corrections, have been published as follows:

Three, those of orders 16 and 96, in Cayley's list mentioned above. The other three were first published in this journal; those of orders 192 and 576,† May, 1893, and the one of order 24, April, 1894.

For degree nine Kirkman again omitted six groups, viz.:

| Order. | Number. | Group. |
|---|---|---|
| 54 | 1 | $(abc\,.\,def)$ cyc $(abc\,.\,ghi)$ cyc $(adg\,.\,bfi\,.\,ceh)$ $\times\,(ad\,.\,be\,.\,cf\,.\,gh)$, |
| | 2 | $[\{(abc)$all$(def)$all$\}$pos$(ghi)$all$]$ sex$(adg\,.\,bfi\,.\,ceh)$<br>or $(abc\,.\,def)$cyc$(abc\,.\,ghi)$cyc$(adg\,.\,bfi\,.\,ceh)$cyc $\times\,(ad\,.\,bf\,.\,ce)$, |
| 72 | 1 | $(abcd\,.\,efgh)$ cyc $(ac\,.\,ef\,.\,gh)(aci\,.\,bgh\,.\,dfe\,.)$ cyc, |
| 108 | 1 | $[\{(abc)$ all $(def)$ all$\}$ pos $(ghi)$ all$]$ sex $\times\,(adg\,.\,bfi\,.\,ceh)$ cyc $(ad\,.\,be\,.\,cf\,.\,gh)$, |
| 162 | 1 | $\{(abc)$ pos $(def)$ pos $(ghi)$ pos $+$ $(abc)$ neg $(def)$ neg $(ghi)$ neg$\}\,(adg\,.\,beh\,.\,cfi)$ cyc, |
| 648 | 1 | $\{(abc)$ all $(def)$ all $(ghi)$ all$\}$pos $\times\,(adg\,.\,beh\,.\,cfi)$ cyc $(ad\,.\,be\,.\,cf)$. |

It is impossible to decide which one of the two groups $54_2$ is given by Kirkman, for they are conformal, and one has 1120 while the other has 3360 conjugates in the symmetric group of

* *Quarterly Journal of Mathematics*, vol. 25, pp. 137–155.

† These two are given in Professor Cole's "Note on the substitution groups of six, seven, and eight letters." This "note" is also found in *Quarterly Journal of Mathematics*, vol. 26, pp. 372–376.

degree nine. Kirkman gives 1680 as the value of $Q$, which is certainly erroneous.

It may be well to state here that numerous errors occur in Kirkman's notation for other groups, especially in the higher orders. He never gives any more groups of a given order than actually exist. It therefore seems quite likely that the errors in notation are due to insufficient study of the groups under consideration. This view is, perhaps, supported by the author's statement in the introduction to the list, viz., "If any (transitive groups whose degree is less than 11) have escaped me, it is the fault, not of my method, but of my carelessness." For degree ten the list is so inaccurate as to be of little value.

In 1872, Camille Jordan gave an enumeration of the numbers of primitive groups for each degree through degree seventeen in *Comptes Rendus,* 75, p. 1757. Only two omissions have been published, one of degree * nine and the other of degree twelve.†

About twenty years later ‡ Askwith, who apparently was unacquainted with Serret's list,§ published two articles in the *Quarterly Journal of Mathematics* ‖ in which he endeavored to give all the groups for each degree from three to nine, together with the methods by which they may be found. He referred only to Serret's work,¶ although two more comprehensive treatises ** had been published.

In the following number of the same journal, Professor Cayley gave the results which Askwith had obtained, together with some additions, in a condensed and perspicuous form.†† In the introduction to this list he explains his notation and calls attention to some very important principles in group construction. The list is very inaccurate. Twenty intransitive and twenty-eight transitive groups were omitted.

Since Cayley's notation is far superior to Kirkman's, and since the corrections to Cayley's list are published in the same notation as the list itself, it may be of especial interest to give an outline of these corrections.

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* This journal, vol. 3, p. 245.

† *Ibid.* vol. 1, 2d series, p. 255.

‡ The most important work in the line of our subject, which was published in this period, is that of VERONESE, *Annali di Mathematica* (2), vol. 11, pp. 93–233. This contains lists of the groups of degrees 2, 3, 4, 5, and 6, on pp. 153, 157, 158, 167, 168, 177–190, 232, 233. Both the transitive and the intransitive groups are enumerated.

§ Cf. *Quarterly Journal of Mathematics*, vol. 24, p. 111.

‖ Vol. 24, pp. 111–167 and 263–331.

¶ *Ibid.* pp. 111, 144, 270.

** JORDAN, *Traité des Substitutions*, 1870, and NETTO, *Substitutionentheorie*, 1882. CAUCHY's work, *Exercices d'analyse*, vol. 3, pp. 151–252 (1844), is not quite as comprehensive as Serret's.

†† *Quarterly Journal of Mathematics*, vol. 25, pp. 71–88 and 137–155.

In the "Note on the substitution groups of six, seven, and eight letters,"* Professor Cole gave all the missing groups except two. Some errors in this note are corrected by its author in the *Quarterly Journal.*† It remains to add that in the enumeration at the end of the note, 199 and 48 should be replaced respectively by 200 and 50. The two missing groups were published separately in this journal.‡

It is very probable that the groups through degree eight have been completely determined. The following enumeration may therefore be accepted with a great deal of confidence as a permanent one.

| Number of letters | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| Number of groups | 1 | 2 | 7 | 8 | 37 | 40 | 200 |
| Transitive groups | 1 | 2 | 5 | 5 | 16 | 7 | 50 |

Through degree five Serret's list of 1850 was complete, as stated before.

The work begun by Askwith in the two articles mentioned above is continued in vol. 26 of the same journal. His attempt to determine all the transitive groups of degree nine was not wholly successful.§

All the groups of degree nine ‖ and the transitives of degrees ten ¶ and eleven have recently been published by Professor Cole, and the intransitives of degree ten ** by the author. From the preceding it will be seen that these enumerations according to degree have been pursued to the following extent. All the possible groups through degree ten, the transitives through degree eleven,†† and the primitives through degree seventeen have been determined.

It should not be inferred that the enumerations according to degree are the only ones which have received attention. In addition to these, there are enumerations of the primitive groups according to class, ‡‡ of simple groups

* This journal, May, 1893; *Quarterly Journal of Mathematics*, vol. 26, pp. 372–376.

† Vol. 27, p. 50.

‡ Vol. 3, pp. 168 and 242–244. Professor Moore informed me that this group had been determined in practically the same manner by Noether, *Mathematische Annalen*, vol. 15, pp. 90–95.

§ Cf. this journal, vol. 2, p. 250.

‖ *Quarterly Journal of Mathematics*, vol. 26, pp. 376–388. Cf. *Ibid.* vol. 27, p. 102, foot-note.

¶ *Ibid.* vol. 27, pp. 39–50. Cf. this journal, vol. 1, 2d series, pp. 67–72.

** *Quarterly Journal of Mathematics*, vol. 27, pp. 99–118.

†† The transitives of degree twelve will appear in vol. 28, *Quarterly Journal.*

‡‡ Jordan, *Comptes Rendus*, vol. 75, p. 1757; Netto, *Theory of Substitutions* (Cole's edition), pp. 133–138.

according to order,* of groups whose orders are composed of given kinds of factors,† of regular groups according to degree, ‡ etc.

*Methods of Group Construction.*

Most of the lists which have been mentioned are preceded by brief explanations of the methods employed by the authors, and references are generally given to works where more complete explanations may be found. Among the separate articles on this subject, special attention should be called to the two following: "On the construction of intransitive groups"§ by Professor Bolza, and "Construction of composite groups" ‖ by Professor Hölder.

The methods of finding all the groups of a given degree may be divided into two great classes: (1) those by means of which we prove the existence of groups and find their distinguishing features, and (2) those by means of which we prove the non-existence of given types of groups in certain regions. The method of the second class employed in the following examples has extensive application in the construction of such non-primitive groups.

In constructing the non-primitive groups of degree ten, it is necessary to inquire whether there is a non-primitive group with five systems of two letters which is simply isomorphic to $(abcde)$ pos. If we interchange the systems only according to the substitutions of the given group in five letters and preserve the same order of the letters of the systems throughout, we evidently obtain $(abcde\,.\,fghij)$ pos. The average number of letters in this group is eight,¶ and the average number in the required group is nine.¶ If such a group exists, it must therefore be possible to solve the Diophantine equation,

$$20\,a + 15\,b = 60 \ (a = 0,\ 2,\ 4;\ b = 0,\ 2).$$

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* Hölder, *Mathematische Annalen*, vol. 40, pp. 55–89; Cole, *American Journal*, vol. 14, pp. 378–388, and vol. 15, pp. 303–316; Frobenius, *Berliner Mittheilungen*, 1893, pp. 221–229; *Sitzungsberichte*, 337–345.

† Netto, *Theory of Substitutions* (Cole's edition), pp. 145–149; Young, *American Journal*, vol. 15, pp. 124–179; Cole and Glover, *ibid.* pp. 191–221; Hölder, *Annalen*, vol. 43, pp. 301–413, and *Göttinger Nachrichten*, 1895, pp. 211–229; Sylow, *Acta Mathematica*, vol. 11, pp. 201–256; Burnside, *Proceedings of the London Mathematical Society*, vol. 25, pp. 9–18.

‡ Kempe, *Philosophical Transactions*, vol. 177, pp. 37–43; Cayley, *American Journal*, vol. 11, pp. 139–157; Levavasseur, *Comptes Rendus*, vol. 120, pp. 822–825 and 899–903; vol. 121, pp. 238–240.

§ *American Journal of Mathematics*, vol. 11, pp. 195–214.

‖ *Mathematische Annalen*, "Bildung zusammengesetzter Gruppen," pp. 321–422. A series of articles on elementary methods of group construction are now being published in the *American Mathematical Monthly*.

¶ Frobenius, *Crelle*, vol. 101, p. 287. Cf. "A simple proof of an elementary theorem in substitutions," etc. This journal, current vol., p. 75.

This equation depends directly upon the fact that the substitutions of the non-primitive group must interchange all the systems represented by the letters of the corresponding substitution of $(abcde)$ pos, and leave the remaining systems unchanged. Since the given Diophantine equation has clearly no solution, no non-primitive group of the required type exists.

Similarly there can be no non-primitive group of degree twelve whose six systems of two letters correspond to the elements of $(abcdef)$ pos and whose base is unity unless the following equation can be solved,

$$40a + 45b + 144c = 360$$

$$(a = 0, 2, 4, 6;\ b = 0, 2, 4;\ c = 0, 2).$$

This has evidently no solution. It is not necessary to examine the corresponding symmetric groups; for, if it is possible that a non-primitive group in the given number of systems is simply isomorphic to either of them, its self-conjugate subgroup corresponding to the alternating group must be intransitive. This could, therefore, be regarded as the base, and its systems of intransitivity as the systems of the required non-primitive group.

It remains to notice a certain dual interpretation of the elements of a non-regular primitive group. If we represent the subgroups which do not contain $a, b, \cdots$ by $A, B, \cdots$, then will every substitution of the group $(abc.de\cdots)$ interchange these subgroups according to an identical substitution in the capital letters $(ABC.DE\cdots)$. We may, therefore, think of the substitutions as interchanging the elements or as interchanging these subgroups. This dual interpretation is not always so interesting in the other groups, since there need not then be a 1, 1 correspondence between the elements and the capital letters.*

Whenever all the capital letters stand for subgroups whose degree is one less than the degree of the group, such a 1, 1 correspondence must exist even if the group is intransitive or non-primitive.

LEIPZIG, *October*, 1895.

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* Cf. MOORE, this journal, vol. 1, 2d series, pp. 61, 62.