ON CAUCHY'S THEOREM CONCERNING COMPLEX INTEGRALS.

BY PROFESSOR MAXIME BÔCHER.

THE following arrangement of the proof of Cauchy's theorem, that the integral of a holomorphic function in a simply connected region is a function of the limits of integration, but not of the path, is more elementary than the proofs ordinarily met with. Like the proofs given by Goursat * and Jordan,† it avoids the use of double integrals or the calculus of variations, and it has the advantage over these proofs not merely of brevity, but also of avoiding more or less complicated considerations of limits. The materials from which the following proof is built up are so familiar to all students of the theory of functions,‡ and the consequent probability that this method of putting them together is not unknown, is so great, that I should hardly have ventured to publish it, had it not seemed to possess peculiar pedagogical advantages. These depend, apart from its elementary character, on the fact that almost all the steps involved establish theorems, or illustrate methods, with which the student must become familiar sooner or later.

The theory of integrals of a function of a complex variable depends, as is well known, on the theory of integrals of the form

$$I = \int_{(a,\,b)}^{(x,\,y)} (Pdx + Qdy),$$

in which P and Q are real functions of the two real variables x, y. We will begin with the theorem

S being any region in the xy-plane in which P and Q are single-valued and have continuous first partial derivatives, a necessary and sufficient condition that the integral I should depend merely on the limits of integration is the existence in S of a single-valued function $\phi(x, y)$, for which $\dfrac{\partial \phi}{\partial x} = P$, $\dfrac{\partial \phi}{\partial y} = Q$.

For, if such a ϕ exists, $Pdx + Qdy$ is the complete differential of ϕ, and the integral I is the limit of the sum of the increments of ϕ, which we get in going along the path of integration from (a, b) to (x, y). But the sum of these increments is $\phi(x, y) - \phi(a, b)$, so that I is independent of the path of integration. Conversely, if I is independent of the path, it

---

* See HARKNESS and MORLEY'S *Theory of Functions*, p. 164.

† *Cours d'Analyse* (2d ed.), vol. 1, p. 185.

‡ See, for instance, PICARD, *Traité d'Analyse*, vol. 1, pp. 81–83, and BOUSSINESQ, *Cours d'Analyse infinitésimal*, vol. 2, pp. 6–12.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

will be itself (the lower limit (a, b) being regarded as constant) a function $\phi(x, y)$ of the sort we want; for, in the first place, I is now single-valued in S by hypothesis. Its increment when x alone changes is

$$\Delta_x I = \int_{x, y)}^{(x+\Delta x, y)} (Pdx + Qdy).$$

In this integral the path of integration may be taken as a straight line, so that y is constant. We get then

$$\Delta_x I = \int_x^{x+\Delta x} P(x, y)dx = \Delta x \cdot P(x + \theta\Delta x, y),$$

where $0 \leqq \theta \leqq 1$. Dividing through by Δx, and taking the limit as Δx approaches zero, we get $\dfrac{\partial I}{\partial x} = P(x, y)$. Similarly, we find $\dfrac{\partial I}{\partial y} = Q(x, y)$.

Obviously, then, a necessary condition that I should be independent of the path is that at every point of S

$$(A) \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

since both sides of this equation will be $\dfrac{\partial^2 \phi}{\partial x \partial y}$. It remains to show that this is also a sufficient condition, provided that the region included between the paths of integration belongs wholly to the region S, as would necessarily be the case if S were simply connected. We first prove the following:

Lemma. — *Equation* (A) *is a sufficient as well as a necessary condition for rectangular regions* S *whose sides are parallel to the axes of* x *and* y.

To establish this lemma, we must show that if equation (A) holds throughout our rectangle, a single-valued function ϕ exists there, such that $\dfrac{\partial \phi}{\partial x} = P$, $\dfrac{\partial \phi}{\partial y} = Q$. Such a ϕ we get here again in the integral I itself, provided that, in order to ensure that I shall be a single-valued function of x, y, we agree to make the path of integration consist of two straight lines, — from (a, b) to (x, b) and from (x, b) to (x, y). Owing to the restricted shape of S, these two lines will always lie wholly in S. We have then

$$\phi = \int_a^x P(x, b)dx + \int_b^y Q(x, y)dy,$$

$$\frac{\partial \phi}{\partial y} = Q(x, y),$$

$$\frac{\partial \phi}{\partial x} = P(x, b) + \int_b^y \frac{\partial Q(x, y)}{\partial x} dy = P(x, b) + \int_b^y \frac{\partial P(x, y)}{\partial y} dy = P(x, y).$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

From the lemma just proved, it follows at once that equation (A) is the necessary and sufficient condition that the integral I should vanish when taken around any closed path which lies within or upon the boundary of a rectangle of the sort we have just considered.

We will now state the final proposition we have to prove in the following form:

Whatever shape the region S may have, the sufficient as well as the necessary condition that the integral I taken in the positive direction around the complete boundary of any region S' lying actually within S should be zero, is equation (A).*

To prove this, let us divide up that portion of S where S' lies into rectangles by lines parallel to the axes of x and y, and take these rectangles so small † that those rectangles which lie partly within and partly without S' shall lie wholly within S. Then S' will be divided into a number of pieces, some of which are rectangles, while the others are parts of rectangles. The integral taken in the positive direction around the complete boundary of S' is obviously equal to the sum of the integrals taken in the positive direction around the boundary of each of the small pieces into which we have divided S'. But each of these last integrals vanishes if equation (A) holds throughout S, since each of the paths lies within or on the boundary of a rectangle of the sort considered in the preceding lemma, throughout which equation (A) holds. The integral taken around the complete boundary of S' will therefore vanish, if (A) holds throughout S.

Bearing in mind that

$$\int (u+vi)(dx+idy) = \int (udx - vdy) + i\int (vdx + udy),$$

we infer at once from the proposition just proved the following proposition:

u and v being single valued real functions of the real variables x, y in a region S of the xy-plane, and having there continuous first partial derivatives, the necessary and sufficient condition that $\int (u+vi)(dx+idy)$ taken around the boundary of a region S' which lies within S should be zero is:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

* That is, the boundary of S' must not coincide with that of S at any point.

† It should be carefully noticed that these rectangles are not "infinitesimal," *i.e.* that we are not going to allow them to decrease indefinitely.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

It might be objected that in the above proof we assume that the region S' around whose boundary we integrate lies within another region S, throughout which the conditions of continuity and the equation (A) are satisfied. In the first place, however, this will be true in all cases to which we ordinarily apply the theorem;* and in the second place the proof can easily be so modified as to obviate this difficulty, at least for all ordinary shapes of the boundary. All we should have to do would be by a slight extension of the method to prove the lemma given above, not for rectangles with sides parallel to the axes of x and y, but for regions bounded by three sides of such a rectangle, and on the fourth side by a curve which is cut by no line parallel to the two parallel straight sides in more than one point. It will clearly be possible, in any ordinary case, to take the rectangles into which we cut up the region S' so small that the pieces of rectangles which occur near the boundary of S' shall be of this nature.

HARVARD UNIVERSITY, *December*, 1895.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, January 25, at three o'clock, the President, Dr. HILL, in the chair. There were seventeen members present. One nomination for membership was received. The report of the auditing committee, appointed at the preceding meeting to examine the Treasurer's accounts, was presented and accepted. The following papers were read:

(1) Professor HENRY S. WHITE: "Kronecker's linear relation among the minors of a symmetric determinant."

(2) Professor H. TABER: "On certain sub-groups of the general projective group."

In the absence of Professor White his paper was read by Mr. Ling.

THE following mathematical courses are offered in the University of Leipzig for the summer semester of the present year: — Professor Scheibner: Theory of numbers; — Professor Neumann: Selected chapters in mathematical physics; — Professor Lie: Theory of groups (continuous transformation groups); — Professor Mayer: Differential equations of dynam-

* The only case when it would not be true would be when the path of integration meets a natural boundary of the function we are integrating.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use