The main question is whether, if $f(x)$ has a derivative,

$$f'(x) = u_1'(x) + u_2'(x) + \cdots$$

is a true equation. The right-hand side can be written in the form

$$\lim_{n=\infty} \left[\frac{d}{dx}\int_0^n f(x,y)\,dy\right] = \lim_{n=\infty}\left[\int_0^n \frac{\partial f(x,y)}{\partial x}\,dy\right]$$

$$= \int_0^\infty \frac{\partial f(x,y)}{\partial x}\,dy,$$

and thus the question is reduced to that of whether

$$\frac{d}{dx}\int_0^\infty f(x,\,y)\,dy = \int_0^\infty \frac{\partial f(x,y)}{\partial x}\,dy$$

is a true equation.

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LINEAR DIFFERENTIAL EQUATIONS.

Einleitung in die Theorie der linearen Differentialgleichungen mit einer unabhängigen Variablen. Von DR. LOTHAR HEFFTER. Leipzig, Teubner, 1894. 8vo, XIV+258 pp.

IN teaching higher mathematics, the question presents itself, to what functions beyond the algebraic and elementary transcendental functions should the student be introduced first? The answer which is given to this question almost as a matter of course is: the elliptic and then the Abelian functions. Without in any way casting doubt upon the wisdom of the choice here expressed for many cases (perhaps even for most cases as far as the elliptic functions go), it may be pointed out that the above is by no means the only satisfactory answer, and that the explanation of its almost universal acceptance is to be found in great part in mere tradition. Another class of functions which forms from many points of view an equally satisfactory introduction to the study of the higher transcendental functions, is the class with which the book under review deals, *i. e.*, functions defined by homogeneous linear differential equations. Not only is this true of the study of these functions from a purely theoretical point of view, but when we look in the direction of applied mathematics, we find more physical problems which lead to functions of this class (for instance, Bessel's functions, zonal harmonics, etc.), than require the use of elliptic integrals or elliptic functions.*

The preference we have noted for elliptic and Abelian functions owes much of its persistence to the excellent and available text-books on the subject. The appearance of a text-book introductory to the theory of linear differential equations must, therefore, be a cause for satisfaction, not only to those who are especially interested in this theory, but to all who are desirous of seeing an equable and not a one-sided development of mathematical instruction and study. It would be especially fortunate if the general introduction of the study of linear differential equations, as a subject coördinate with and fully as important as the study of elliptic and Abelian functions could help to lessen the estrangement, which now exists between pure and applied mathematicians. We fear, however, that the book under review can contribute only very indirectly to this most desirable result, as the author not only considers no applications, but does not present even the theory in such a way as to be most useful to the student of applied mathematics.

In the execution of his plan of writing an introductory text-book on the subject of linear differential equations, which shall require for its reading as little preliminary training and knowledge as possible, Professor Heffter has been in various respects remarkably successful. The subjects which he has chosen he has treated in an elementary and easily intelligible manner. The arrangement is in many ways a truly admirable one for purposes of instruction, as the direct formal treatment of certain important special cases, not unlike the methods with which the student is familiar in more elementary mathematics, is made to precede and lead up to the more abstruse methods of the theory of functions. Moreover, from time to time, the course of the discussion is interrupted by a concise but lucid recapitulation of the main results arrived at, and a sketch of the lines along which the further developments are to proceed. Such illustrative examples as are given are well chosen, though the book does not come up either in number or variety of examples to the English standard.

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* This comparison will be still more striking, if we remember that the so-called "complete" elliptic integrals regarded as functions of the modulus belong quite as much to one of the above classes as to the other, since they satisfy certain linear differential equations of the second order.

While recognizing the above excellent features, it is necessary for us to insist upon one great defect; the narrow choice of subject matter. The author's own strong interest in certain questions has made him blind to the needs of the student. If the book were intended for specialists this would be a comparatively unimportant point; the writer of a monograph has the right to choose his own subject. But as has been pointed out the book under review aims to be and is essentially a text-book, and we may fairly insist that a text-book should present the different important sides of the elements of the theory. In the volume before us, however, many important and elementary matters are crowded out by the detailed development of a single subject with all the minuteness of which it is susceptible. This criticism will be better understood after we have summarized the contents of the book.

The volume opens with a six-page Introduction. The general form of the differential equations to be considered is first given, namely:

$$p_0(x)\frac{d^ny}{dx^n}+p_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+p_n(x)y+p(x)=0,$$

and it is explained that the coefficients $p_0(x)$, $p_1(x)$ $\cdots\cdot$ will be regarded as given throughout a part (or the whole) of the x-plane as singled valued analytic functions of the complex variable x. The book deals almost exclusively with *homogeneous* equations so that $p(x)$ is usually identically zero. The idea of singular points of the differential equation is explained and the Introduction closes with a general sketch of the problems to be considered.

The first three chapters are devoted to the question of obtaining solutions of a homogeneous linear differential equation in the form:

$$y=(x-a)^{\alpha}[c_0+c_1(x-a)+c_2(x-a)^2+\cdots\cdot]$$

when a is a point at which none of the coefficients of the differential equation has an essentially singular point. In the first chapter, this series is substituted into the differential equation, and an infinite set of equations is obtained for determining the constants α, c_0, c_1, $\cdots\cdot$. In Chapter II. these equations are solved, the number of different series of the above form which can be found being carefully discussed for all possible cases. At the beginning of the third chapter, it is pointed out, that although the series thus found *formally* satisfy the differential equation, they will not always converge. We, therefore, restrict ourselves to a special class of points, the *regular* points,* for which all the series which can be formed converge. This convergence is established by the method originally used by Fuchs. We are at a loss to understand why the far shorter and equally elementary method of Frobenius† was not adopted.

Inasmuch as the non-singular points are special cases of the regular points, the fundamental theorem concerning the existence of solutions in the neighborhood of non-singular points is here obtained as a special case.‡

In Chapter IV. the chief subjects taken up are the linear dependence of solutions; the method of obtaining n linearly independent solutions when once we assume that we can find one solution of any homogeneous linear differential equation of order not higher than n; and the reduction of non-homogeneous to homogeneous equations.

Chapter V. involves some simple applications of the previous results to non-singular and apparently singular points, and a treatment of equations with constant coefficients.

In Chapter VI. the analytic continuation of the solutions is considered, and the idea of the group of the differential equation is explained.

We have as yet succeeded only "in general" in developing n linearly independent solutions about a regular point,

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*Although this class of points was systematically discussed by Fuchs in 1866, no name was given to them, until 1873 when Thomé introduced the term *regular*, a term which has been adopted by Frobenius, Klein and others in Germany as well as by French and English writers. Unfortunately Fuchs in 1886 introduced the term *Stelle der Bestimmtheit* which has been adopted by Heffter and Schlesinger. The objection to the word regular seems to have been that Weierstrass, not so much in his work on differential equations as in the general theory of functions, had used the word in the sense of non-singular. The reviewer would suggest, however, that no inconvenience would arise if the use of the word regular in Weierstrass's sense was abandoned. We speak of a function as being analytic throughout a given region, if it can be developed into a power series about every point of the region. Why should we not speak of a function as analytic at a point if it can be developed into a power series about this point? We should then of course have to distinguish between the phrase "analytic *at* a point" and "analytic *in the neighborhood* of a point."

† Crelle, Vol. 76, pp. 218–219. Reproduced in Craig's *Treatise*, Vol. I., pp. 93–98, and Schlesinger's *Handbuch*, Vol. I., pp. 164–168. The portion of the proof relating to uniform convergence does not of course concern us here.

‡ Many readers will think that this fundamental theorem should be brought in earlier instead of not being even mentioned until p. 41. The reviewer is inclined to agree with this view in spite of the fact that the arrangement here adopted makes it possible to avoid certain repetitions, in particular the necessity of giving two proofs of convergence.

since when two or more roots of the indicial equation differ by an integer, there will in general be only a smaller number of solutions developable in the form above used. The next three chapters are devoted to the discussion of these special cases.

In Chapter VII. it is shown how in all these cases n linearly independent solutions can be developed about a regular point, if we introduce logarithmic terms into the developments. In Chapter VIII. the actual formulæ for computing the coefficients in these developments are discussed at great length, while in Chapter IX. the question is considered how the solutions we have obtained involving a number arbitrary constants, can be specialized by the choice of these constants, so as to be in some respect peculiarly simple.

The chief result we have obtained up to this point is that n linearly independent solutions can be developed about regular points in certain definite forms. In Chapter X. it is shown that it is *only* in the case of regular points that this can be done.

The next three chapters are devoted to the discussion of irregular points. In the first of them the linear substitutions are discussed, to which a set of n linearly independent solutions are subjected, when we follow them around a singular point, the *fundamental equation* of this point is introduced and discussed, and certain applications are made to the case of regular points particularly to the subjects treated in Chapter IX. In the next chapter, the development of n linearly independent solutions about an irregular point is considered, the last part of the chapter being devoted to the extension of the results of Chapter IX. to the case of irregular points. In the last of these three chapters the case of an irregular point is discussed at which some of the integrals are regular. This question is made to depend upon the idea which is here introduced of the *reducibility* of a differential equation. As the student has no means of determining whether a given differential equation is reducible or not, the subject of this chapter does not seem to have been very happily chosen for an elementary treatise.

In Chapter XIV. the point at infinity is considered, and in Chapter XV. differential equations are briefly treated, all of whose singular points are regular.

The volume closes with an appendix of twenty-three pages, in which a number of theorems are proved to which reference is made in the body of the book. These theorems are such as could not fairly have been assumed to be known, although they are not necessarily connected with the subject of differential equations.

Omitting the appendix, the volume before us may be roughly divided as follows: 147 pages devoted directly to the development of solutions about regular points,* and 85 pages of other matter. These figures show the preponderance of a single subject over all other subjects combined. The development about regular points is certainly an important question, but no one familiar with the theory of linear differential equations will attribute to it such an overshadowing importance as the space given to it in this book would seem to indicate. In a text-book, however, it is not merely the intrinsic value of a subject which should determine the weight to be given it, its educational value may also well be considered. In this respect the theory of regular points is useful as an introduction to more intricate questions, but beyond this its value is very slight, as the methods used are chiefly of a formal nature, not very different from those with which the student is already familiar. What makes the disproportionate attention here given to regular points still more to be regretted, is that a treatment sufficient for the wants of a beginner need not be very long. It is the discussion of special cases which make the presentation difficult, and although it is important for the reader to understand these special cases in a general way, the detailed discussion here given is not only of relatively slight importance, but (and this is an almost fatal objection in a text-book), absolutely uninteresting to most readers.†

In spite of the defect upon which we have just dwelt, we believe that the book can, especially under the guidance of a judicious teacher, be satisfactorily used as an introductory text-book. Certain portions, for instance Chapters VIII. and IX., may be wholly omitted, while in other

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* This includes Chapters I.–III., V., VII.–X., XIV, XV. Portions of these chapters refer to irregular points, but this fact is about made up for by the sections on regular points in the other chapters.

† One way of avoiding these difficulties in teaching is to confine oneself to equations of the second order (which are after all by far the more important, both theoretically and practically), leaving perhaps the student to work out some of the corresponding properties of equations of the n^{th} order. The number of special cases for equations of the second order is very small, but not too small to be fairly typical of the general case. Cf. the pamphlet by the writer recently published by Harvard University: *Regular Points of Linear Differential Equations of the Second Order.* In this pamphlet about half the matter in the volume under review is treated, in so far as it relates to differential equations of the second order, in the space of twenty-two small pages.

parts, as in Chapter II., whole pages, and those the hardest pages to read, can be replaced by a very short explanation on the part of the teacher. If the time spent upon the text-book is thus abbreviated, it would be possible even in a short course to go on to other questions, both more instructive and more interesting than the formal matters thus omitted, such for instance as Schwarz's s-functions, the real solutions of real differential equations, or the study of irregular points by means of infinite determinants or semiconvergent series.

Although a considerable list of misprints has been noted in a page of Errata, placed after the table of contents, several others have escaped notice. This list of Errata includes besides actual misprints, the correction of a few more or less trifling mistakes. There are unfortunately certain mistakes which even here have escaped the author's notice. One which has been transcribed directly from Fuchs's memoir in Crelle, vol. 66, p. 150, occurs near the bottom of p. 37. There will be in general no real positive quantities $M_1 \cdots M_n$ greater than the absolute values of the quantities (7), throughout the circle of radius r, since the quantities (7) will in general become infinite at some point of this circle. It is absolutely necessary here to introduce a second circle with a radius a little smaller than r. A second error occurs near the end of § 46. The "neighborhood" of the point $x=0$ for the equation (1′) is not U as is here stated, but in general, smaller than U. That the series in formula (8^a) p. 89, nevertheless converge throughout U requires of course a proof which is not there given, but which can be easily supplied.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
September, 1896.

NOTES.

A SPECIAL Meeting of the American Mathematical Society was held at Princeton University, on Saturday, October 17, at quarter past three, P. M. There were thirty-four members of the Society and thirteen visitors present. The President, Dr. G. W. HILL, occupied the chair, and introduced Professor FELIX KLEIN and Professor J. J. THOMSON, who addressed the Society. Professor KLEIN discussed the stability of a sleeping top. Professor THOMSON spoke