NOTES ON THE THEORY OF BILINEAR FORMS.

BY PROFESSOR HENRY TABER.

[Read at the November Meeting of the Society, 1896.]

ASSOCIATED with any linear substitution

$$x_r = \sum_1^n {}_s\, a_{rs}\, y_s \qquad (r=1, 2, \cdots u)$$

is the bilinear form

$$\sum_1^n {}_r \sum_1^n {}_s\, a_{rs}\, x_s\, u_r,$$

which we may regard as representing the substitution; and in the symbolic system described below we need not distinguish between a linear substitution and the bilinear form associated with it. Thus we may use the same symbol A to denote indifferently the substitution and the bilinear form.*

If B denotes the bilinear form $\sum_1^n {}_r \sum_1^n {}_s\, b_{rs}\, x_s\, u_r$, $A \pm B$ denotes either the bilinear form $\sum_1^n {}_r \sum_1^n {}_s\, (a_{rs} \pm b_{rs})\, x_r\, u_r$, or the linear substitution to which this form corresponds. Further, in this symbolic notation, $A\,B$, which is termed the product of A into B, signifies either the bilinear form $\sum_1^n {}_r \sum_1^n {}_s\, c_{rs}\, x_s\, u_r$, where

$$c_{rs} = \sum_1^n {}_t\, a_{rt}\, b_{ts} \qquad (r, s = 1, 2, \cdots n),$$

or its associated linear substitution, which is obtained by the composition in a definite order of the substitutions A and B. The composition or "multiplication" of bilinear

* In the notation and nomenclature invented by Cayley a linear substitution and a bilinear form are each represented by the square array of its coefficients, the *matrix* of the bilinear form or of the linear substitution; and we do not distinguish in general between a bilinear form and its matrix, or between a linear substitution and its matrix. See "Memoir on the Theory of Matrices," also "Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function," *Philosophical Transactions*, 1858. Cayley's symbolic notation differs only in unessential features from the system here described, which was invented by Frobenius subsequent to Cayley's investigations.

forms is then associative and distributive, but not in general commutative.

In this notation, the equation $A = B$ is to be regarded as satisfied identically, that is to say, for all values of the x's and u's, and is thus equivalent to n^2 equations between the corresponding coefficients of A and B. In what follows I shall denote the identical substitution, or its associated bilinear form $\sum_1^n {}_r x_r u_r$, by I. The determinant of the linear substitution or bilinear form A will be denoted by $|A|$. If $|A| \neq 0$, there is a definite form, the *reciprocal* of A, denoted by A^{-1}, such that $A A^{-1} = A^{-1} A = I$.

§ 1.

Relation between the exponents of the elementary divisors of $|A - \rho B|$ *and the numbers belonging to the roots of the equation* $|A - \rho B| = 0$.

Let ρ be a variable parameter not contained in the coefficients of A and B, the host, $A - \rho B$, of bilinear forms is then a sheaf of bilinear forms, and the determinant $|A - \rho B|$ of this sheaf is a rational integral function of ρ of order n.* Let ρ_0 be a root of the equation $|A - \rho B| = 0$; then, if $a - \rho_0 b = 0$, $a - \rho b$ is a divisor of the determinant $|A - \rho B|$. If ρ_0 is a root of multiplicity l of the above equation, $|A - \rho B|$ contains $a - \rho b$ to the power l, but to no higher power. The minors of the determinant $|A - \rho B|$ of all orders are also rational polynominals in ρ; and all the minors of order $n - r$ may contain $a - \rho b$ if $l > r$. Let l_r be the highest power of $a - \rho b$ contained in all the minors of order $n - r$. If l_ν is the last of the series l_1, l_2, etc., that is not zero, we have

$$l > l_1 > l_2 > \ldots. > l_\nu > 0;$$

and, if we put

$$e = l - l_1, \; e_1 = l_1 - l_2, \; \cdots \; e_{\nu-1} = l_{\nu-1} - l_\nu, \; e_\nu = l_\nu,$$

then

$$e \geqq e_1 \geqq e_2 \geqq \cdots \geqq e_\nu \geqq 1,$$

$$(a - \rho b)^l = (a - \rho b)^e (a - \rho b)^{e_1} (a - \rho b)^{e_2} \cdots (a - \rho b)^{e_\nu}.$$

The several factors $(a - \rho b)^e$, $(a - \rho b)^{e_1}$, etc., of $(a - \rho b)^l$ are *elementary divisors* (*elementar theiler*) of the determinant $|A - \rho B|$. These elementary divisors all vanish for $\rho = \rho_0$.

* It is assumed that $|B| \neq 0$.

Corresponding to each root of the equation $|A - \rho B| = 0$ is a system of elementary divisors which all vanish if ρ be equal to the root in question. The theory of the elementary divisors of $|A - \rho B|$ is due to Weierstrass.

Associated with the roots of the equation $|A - \rho B| = 0$, are certain other numbers which have been studied by Sylvester, Buchheim, Ed. Weyr, and others. These numbers play an important rôle in the theory of linear substitution, and are closely related to the exponents of the elementary factors. Thus, ρ_0 being a root of multiplicity l of the equation $|A - \rho B| = 0$, the nullity* of the bilinear form $A - \rho_0 B$ is at least one, and cannot exceed l; and the nullity of successive powers of $A - \rho_0 B$ increases until a power of exponent $\mu \leqq l$ is attained whose nullity is equal to l. The nullity of the $(\mu + 1)^{\text{th}}$ and higher powers of $A - \rho_0 B$ is then also l; and if

$$m_1,\ m_2,\ \cdots\ m_{\mu-1},\ m_\mu = l,$$

denote respectively the nullities of

$$(A - \rho_0 B)\ \ (A - \rho_0 B)^2,\ \cdots\ (A - \rho_0 B)^{\mu-1},\ (A - \rho_0 B)^{\mu},$$

we have

$$m_1 \geqq m_2 - m_1 \geqq \cdots \geqq m_\mu - m_{\mu-1} \geqq 1.$$

I term the numbers m_1, m_2, etc., the numbers *belonging* to the root ρ_0 of the equation $|A - \rho B| = 0$.

Let now a diagram be formed corresponding to ρ_0 by arranging $m_\mu = l$ dots in rows and columns so that there shall be μ rows, respectively, of

$$m_1\,,\ m_2 - m_1\,,\ \cdots\ m_\mu - m_{\mu-1}\,,$$

equidistant dots, and so that the first dot in each row falls in the same (left hand) column. The number of columns will then be m_1, which will be found to be equal to $\nu + 1$; and the number of dots in the successive columns, beginning with the extreme left hand column, will be equal, severally and respectively, to $e, e_1, e_2, \cdots e_\nu$.†

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* The nullity of a bilinear form is m if the $(m-1)^{\text{th}}$ minors (the minors of order $n - m + 1$) of its matrix are all zero, but not all the m^{th} minors (the minors of order $n - m$).

† *E. g.*, if the numbers belonging to ρ_0 are 5, 9, 13, 16, the diagram corresponding to ρ_0 is

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and we have, $e = 4$, $e_1 = 4$, $e_2 = 4$, $e_3 = 3$, $e_4 = 1$.

Thus knowing the numbers belonging to the root ρ_0 of the equation $|A - \rho B| = 0$, we can find the exponents of the elementary divisors of $|A - \rho B|$ that vanish for $\rho = \rho_0$, and conversely. Further, from the existance of certain relations between the numbers belonging to ρ_0, we may show the existance of corresponding relations between the exponents e, e_1, etc. For example, if $\mu = 1$, in which case $m_1 = l$, the e's are all equal to unity, that is the elementary divisors that vanish for $\rho = \rho_0$ are linear (or *simple*); and conversely. Again if the numbers belonging to ρ_0 are all even, ν is odd, and

$$e = e_1, \; e_2 = e_3, \; e_4 = e_5, \cdots e_{\nu - 1} = e_\nu ;$$

that is, the elementary divisors $(\rho_0 - \rho)^{e_r}$ occur in pair with equal exponent. Conversely, if this condition is satisfied, the numbers belonging to ρ_0 are all even.

If B is the identical substitution, or the bilinear form $I = \sum_1^n {}_r x_r u_r$, the determinant $|A - \rho I|$ is the *characteristic function* of A, and the equation $|A - \rho I| = 0$, the *characteristic equation* of A.

§ 2. *Special linear homogeneous group.*

Let $|A| = +1$, the linear substitution A is then a transformation of the special linear homogeneous group. Let the roots of the characteristic equation of A, $|A - \rho I| = 0$, be

$$\rho' = R' e^{\theta' \sqrt{-1}}, \quad \rho'' = R'' e^{\theta'' \sqrt{-1}}, \text{ etc.},$$

respectively, of multiplicity m', m'', etc.,—R', R'', etc., being the moduli, and θ', θ'', etc., the smallest positive arguments of ρ', ρ'', etc. Then since $|A| = +1$,

$$(R')^{m'} (R'')^{m''} \cdots = +1.$$
$$m' \theta' + m'' \theta'' + \cdots = 2 k \pi,$$

where k is some positive integer. Let now δ be the greatest common divisor of the exponents of all the elementary divisors of the characteristic function of A, then by virtue of what was shown in § 1, it follows from a theorem given in the BULLETIN for April, 1896, page 232, that *A can be generated by the repetition of an infinitesimal transformation of the special linear homogeneous groap if, and only if, k contains δ.*

§ 3.

Group of linear transformations whose invariant is a bilinear form.

1. Let G denote the group of linear transformations whose invariant is the form A of non-zero determinant, the x's and u's being cogrediant. Let $\breve{B}$ denote the bilinear form $\sum_r^n \sum_s^n b_{sr} x_s u_r$ conjugate (or transverse) to $B = \sum_r^n \sum_s^n b_{rs} x_s u_r$. Then, if the linear substitution associated with the form B is a transformation of group G,

$$\breve{B} A B = A,$$

and conversely. From what was stated in the introduction, it is evident that this identity gives rise to n^2 equations in the coefficients of A and B which are quadratic in the coefficients of B. Consequently, the problem to determine the coefficients of the automorphic linear transformation of a bilinear form presents itself at the outset as the solution of a system of n^2 quadratic equations in n^2 quantities (the coefficients of B). The reduction of the general solution of this problem to the solution of a system of n^2 equation linear in n^2 variables was effected by Cayley in 1858. Thus Cayley showed that if the linear substitution B is a transformation of group G, we may in general put

$$B = (A + C)^{-1} (A - C),$$

where C is a bilinear form satisfying the equation

$$(a) \qquad A^{-1} C + (\breve{A}^{-1}) \breve{C} = 0.*$$

As shown by Cayley, this equation is satisfied identically if A is symmetric, provided C is alternate; or if A is alternate, provided C is symmetric.† A form A is, of course, symmetric if $\breve{A} = A$ (*i. e.*, if $a_{sr} = a_{rs}$), and is alternate if $\breve{A} = -A$ (*i. e.*, if $a_{sr} = -a_{rs}$).

The next step in Cayley's investigation was, if A is neither symmetric nor alternate, to express the coefficients of C (and hence of B) rationally as functions of the smallest

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* *Philosophical Transactions*, 1858, p. 44.

† These two theorems are almost invariably ascribed by German mathematicians to Frobenius, by whom they were indeed given in 1878 (*Crelle*, vol. 84); but they had been given by Cayley twenty years earlier in the *Philosophical Transactions.*

number of parameters; but, as pointed out by Voss, Cayley's solution of this latter problem is erroneous. The determination of the bilinear form C satisfiying equation (α) is due to Voss.

If -1 is a root of the characteristic equation of B, this linear substitution cannot be given Cayley's representation.

In the *Mathematische Annalen*, vol. 48, p. 102, Mr. Loewy states that if A is a bilinear form (of non-zero determinant), whose automorphic linear transformations of determinant $+1$* form an irreducible system,† every proper transformation of group G results from the composition of two of Cayley's forms, that is, can be given the representation

$$\beta) \qquad (A + C_1)^{-1} (A - C_1) (A + C_2)^{-1} (A - C_2),$$

where C_1 and C_2 satisfy equation (α).

I find that for certain forms A (of non-zero determinant), for which $|A - \breve{A}| = 0$, not every automorphic transformation of determinant $+1$ can be given this representation. But, if A is any bilinear form whatever of (non-zero determinant), such that neither the determinant $|A + \breve{A}|$ nor the determinant $|A - \breve{A}|$ is zero, every automorphic linear transformation can be given Mr. Loewy's representation, that is, results from the composition of two of Cayley's forms.

In this case, that is, if $|A \pm \breve{A}| \neq 0$, every transformation of group G is given by the second power of Cayley's expression, that is, may be given the representation

$$(\gamma) \qquad [(A + C)^{-1} (A - C)]^2,$$

where C satisfies equation (α). For, in a paper read before the Society, August, 1895, (to appear in the *Quarterly Journal*), I showed that, if neither $|A + \breve{A}|$ nor $|A - \breve{A}|$ is zero, every transformation of group G is the second power of a transformation of this group;‡ and it is readily proved that the second power of a transformation of group G is given by the square of Cayley's expression.

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*If B is a transformation of group G, $|B|^2 = 1$. If A is alternate, all the transformations of group G are proper, that is are of determinant $+1$.

†See Voss. Abhandlungen der k. bayer. Akadamie der Wiss., II. Cl., XVII. Bd., II. Abth., 1890.

‡See BULLETIN, for October, 1895, p. 5. I also showed in this paper that, if A is alternate, every orthogonal substitution of group G is the second power of an orthogonal substitution of this group, and can therefore be generated by the repetition of an infinitesimal orthogonal substitution of group G.

2. Every transformation of group G that can be given by Cayley's representation may be generated by the repetition of an infinitesimal transformation of group G. *Afortiori*, every transformation of group G that can be given the representation (γ), and, therefore, the second power of any transformation of group G, can be generated thus.* It is, therefore, of some importance to determine the conditions necessary and sufficient that a transformation can be expressed by formula (γ).

For the case in which A is symmetric, I have shown in the *Proceedings of the London Mathematical Society*, vol. 26, p. 364, *et seq.*, that in order that a transformation B of group G may be the second power of a transformation of this group, it is necessary and sufficient that either -1 shall not be a root of the characteristic equation of B, or that the numbers belonging to -1 shall all be even. By what was shown in § 1 it follows that this theorem is equivalent to the following: If A is symmetric, a transformation B of group G is the second power of a transformation of this group if, and only if, the elementary factors of the characteristic equation of B that vanish for $\rho = -1$ occur in pairs with equal exponent; provided such factors exist, otherwise no condition is necessary. In this form the preceeding theorem was given by Mr. Loewy, subsequent to my article, in the number of *Annalen* above referred to.

For the case in which A is alternate, I showed in vol. 46 of the *Mathematische Annalen*, p. 582, that if -1 is a root of the characteristic equation of the transformation B of group G, the numbers belonging to -1 are all even. In the paper above referred to, read before the August meeting of the Society, 1895, I showed that these conditions were also sufficient.† This theorem was communicated to the *American Academy of Arts and Sciences*, January, 1896.‡

Therefore, by § 1, if A is alternate, the necessary and sufficient conditions that a transformation B of group G shall be the second power of a transformation of this group are that either -1 shall not be a root of the characteristic equation of B, or, if -1 is a root of the characteristic equation of B, that the elementary factors of the determinant $| B - \rho I |$ that vanish for $\rho = -1$ shall occur in pairs with equal exponent.

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*See *Proceedings American Academy of Arts and Sciences*, vol. 30, p. 181; also paper on "The Group whose Invariant is a Bilinear Form," to appear in a forthcoming number of the *Mathematical Review*.

† See BULLETIN for October, 1895, p. 5.

‡ See *Proceedings*, vol. 31, p. 349.

Since, if $| A \pm \breve{A} | \neq 0$, each transformation of group G is the second power of a transformation of this group, it follows, in this case, that if -1 is a root of the characteristic equation of a transformation of this group, the numbers belonging to -1 are all even.*

3. Let it be assumed that $| A \pm \breve{A} | \neq 0$; and let the linear substitution B belong to the sub-group of orthogonal substitutions of group G. If -1 is not a root of the characteristic equation of B we may put

$$B = (I + C)^{-1} (I - C),$$

where C is alternate, and is commutative with A. From the theorem given in the foot-note p. 161, it follows that any orthogonal substitution B of group G may be given the representation

$$[(A + C)^{-1} (A - C)]^2,$$

where C satisfies equation (α).

§ 4.

Group whose invariant is a real quadratic form.

If we put $u_1 = x_1$, $u_2 = x_2$, etc., A becomes a quadratic form. In this case we may assume that A is symmetric.

Let now G denote the group of real linear substitutions whose invariant is the quadratic form A. For $n = 4$, I have shown in the *Proceedings of the London Mathematical Society*, vol. 26, p. 375, that there are proper transformations of group G that are not the second power of any transformation of this group, and which, therefore, cannot be generated by the repetition of an infinitesimal transformation of this group, provided the roots of the characteristic equation of A are not all of the same sign. Whence, in this case, it follows, if $n > 4$, that there are proper transformation of group G not the second power of any transformation of the group, and for which, therefore, the elementary divisors that vanish for $\rho = -1$ do not all occur in pairs with equal exponent.†

On the other hand, if A is real, and if the roots of the

* The same is true, in this case, of the numbers belonging to the root $+1$ of the characteristic equation of a transformation of group G.

† Mr. Loewy states that, if A is real, and if -1 is a root of a transformation of group G, the elementary divisors $(1 + \rho)^{e_r}$ are necessarily all linear, and that therefore every proper transformation of group G can be given the representation of formula (γ).

characteristic equation of A are all of the same sign, in particular, if $A = I$, in which case G is the group of real orthogonal substitutions, every proper transformation of group G is the second power of a transformation of this group. Therefore, even when A is real, if the roots of the equation $| A - \rho I | = 0$ are not all of the same sign and $n \geqq 4$, G is not isomorphic with the group of real orthogonal substitutions.

NOTES.

THE Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Wednesday afternoon, December 30, at three o'clock, the President, Dr. HILL, in the chair. There were twenty-four members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Professor THOMAS WILLIAM EDMONDSON, New York University; Professor JAMES LAWSON PATTERSON, Union College, Schenectady, N. Y. Reports were presented by the Secretary, Librarian and Treasurer. The Secretary announced that the membership of the Society was 279, a net increase of 12 for the year. The number of new members admitted during the year was 22, and the number of withdrawals 10. The average attendance at the ordinary meetings during the year was 15; the attendance at the last annual meeting 25; at the summer meeting 30; at the colloquium 16; and at the Princeton meeting 34. The number of persons who had attended at least one meeting during the year was 82, an increase of 21 over the preceding year. Mr. Pfister and Dr. Schultze were appointed a committee to audit the Treasurer's accounts.

The chair appointed Dr. Ling and Mr. Pfister tellers for the annual election. Upon canvassing the ballots cast in person and by mail, they announced that the following ticket had been elected:

President,	Professor SIMON NEWCOMB,
Vice-President,	Professor R. S. WOODWARD,
Secretary,	Professor F. N. COLE,
Treasurer,	Professor HAROLD JACOBY,
Librarian,	Professor POMEROY LADUE.

Committee of Publication:

Professor THOMAS S. FISKE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.