KIEPERT'S DIFFERENTIAL AND INTEGRAL CALCULUS.

Grundriss der Differential- und Integral-Rechnung. Von Dr. Ludwig Kiepert, Professor der Mathematik an der technischen Hochschule zu Hannover.

Teil I: *Differential Rechnung.* Siebente vollständig umgearbeitete und vermehrte Auflage des gleichnamigen Leitfadens von weil. Dr. Max Stegemann. 1895. Pp. 638.

Teil II: *Integral-Rechnung.* Sechste vollständig umgearbeitete und vermehrte Auflage, u. s. w., 1896. Pp. 599.

In his Evanston colloquium Professor Klein called the attention of American teachers to this treatise upon the Differential and Integral Calculus as one of the best introductory text-books in the German language. The appearance of the present edition is confirmatory evidence of the high esteem in which it is held by German mathematical teachers. It was originally written by Dr. Max Stegemann, professor of mathematics in the *technischen Hochschule* at Hannover and first appeared in 1862. Since his death the revision of the fifth and subsequent editions has been in charge of his successor, Ludwig Kiepert. The modifications and extensions which have chiefly been made in the interest of students at the universities and technical schools, are so numerous that the original outlines can now scarcely be discerned. The present edition therefore very properly appears under Kiepert's own name.

To American mathematicians the chief interest of the treatise will be due to its German authorship. In reviewing, it will be my aim to bring out those points in which it differs from text-books on calculus in current use in this country, which, when not of English authorship, are generally constructed upon English or French models.

The general range of topics treated in the two volumes is about the same as in our more advanced text-books. The closest parallel is to be found perhaps in Williamson's Differential and Todhunter's Integral Calculus. The standpoint assumed in the discussion is, however, more elementary. The most noticeable variations in subject matter are, on the one hand, the insertion in the differential calculus of a final chapter upon the complex variable and in the integral calculus an introduction to the theory of differential equations; on the other hand, the exclusion of such topics as the gamma-functions, the theory of probabilities and the calculus of variations. As the treatise is professedly a *Leitfaden*, the selection is unquestionably a judicious one. Nothing better could be asked for than the short, pithy chapter upon the complex variable. It is one of the happiest hits of the work. It culminates in De Moivre's formulæ and their applications. I know, indeed, of no place where the comprehensive character of the formula

$$e^{x+iy} = e^{x} (\cos y + i \sin y)$$

as an epitome of the ordinary trigonometric and logarithmic theories is so admirably set forth. The 70-page introduction is also excellent and has the merit of being in touch with the modern development of the subject. Thus, almost at the start it is proved that if $\varphi (x, y)$ and its first partial derivatives are continuous functions of x and y in the vicinity of the point (a, b), the differential equation $\frac{dy}{dx} = \varphi (x, y)$ has in that vicinity a continuous integral of the form

$$y = f(x) = f(a) + f'(a)\cdot(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots.$$

The values of $f'(a), f''(a) \cdots$ are given in terms of the derivatives of φ at (a, b), and the convergence of the series is strictly demonstrated. A similar theorem is established for a system of simultaneous equations

$$\frac{dy}{dx} = \varphi_1(x, y_1, \cdots y_m), \cdots \frac{dy_m}{dx} = \varphi_m (x_1 y_1, \cdots y_m).$$

The author then proceeds to consider in the customary manner various cases where the integration of a differential equation can be effected by means of known functions and its solution given in "*endlicher geschlossener Form.*" He closes with a brief treatment of the ordinary linear equation and with its integration when the coefficients are constant. The modern standpoint is here conspicuous. Thus, to cite a single instance, the general integral of the homogeneous linear differential equation is not simply stated to be

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where $y_1, y_2, \cdots y_n$ are any n particular integrals, but the requirement is made that $c_1, \cdots c_n$ permit of giving to y and its first $n - 1$ derivatives any arbitrary values.

Considerable matter, which with us is relegated to special treatises on algebra, is incorporated into the differential calculus, as for example, the algebraic derivation of the exponential and logarithmic series, chapters on determinants and on convergence and divergence of series, and so on. The plan is in striking contrast with that of our English textbooks which pursue each branch of mathematics as an art and aim in itself. The correlation of the various branches is thus neglected. Often the component parts of even a single branch are not given their proper setting, and the student is consequently at a loss to know what are the important ideas and theorems. So admirable and artistic a work as Salmon's *Conic Sections* will serve as an illustration of what I mean. As a compendious treatment of the subject this treatise is without a rival. Nowhere are the modern geometrical methods applied to the subject with more elegance, but for an introduction of the student to these methods the work is very unsuitable. What student would gather from it any adequate idea of the important metrical properties of the circular points or of the general nature of line coördinates, would dissociate the principle of duality from reciprocation with respect to a conic, or would comprehend in the remotest degree the chapter upon projection, where unreal projections are used without any analytical basis? These and other points are subordinated entirely to the conic section and need disentanglement. The book is therefore chiefly useful after the topics have been studied elsewhere. In Kiepert's treatise, on the other hand, the correlation of parts is carefully attended to. The book is, what it purports to be, a guide to the student. The algebra and the calculus reinforce each other. Each algebraic section receives an independent, though rapid, development, and its interest is heightened by immediate application. The subject of infinite series, commonly the *bête noire* of the student whose introduction to it is made in treatises on algebra, is thus made fresh and interesting. Maclaurin's and Taylor's theorems are first demonstrated, and the practical utility of the series developed by their aid is at once shown by numerous computations of trigonometric functions, of logarithms, and of the value of π. The question of the permissibility of expanding a function into a series then conducts to a discussion of convergence and divergence. The insertion of such algebraic sections is also to be commended on another ground. The limited space forbids the development of any but the most important theorems. These not only serve as a basis for more extended study, whenever it may be needed, but also suffice as a preparation for elementary courses in such subjects as projective geometry and theory of functions, subjects which are too often regarded as beyond the range of college mathematics. In a field so limitless as modern mathematics the saving of time to our young students by the elimination of the relatively unessential is a matter of prime importance. Many an American student abroad feels that he has been placed at a disadvantage in this particular.

Another noticeable feature of the work, and one which is distinctively German, is the emphasis placed upon theory rather than upon the solution of problems. The treatise is not a quarry of examples. There are practically none except those which are worked out, but of these there are many. The confessed aim is to enable the student to acquire by himself, independently of a teacher, a grip upon the calculus, but it surely may be questioned whether this can be accomplished when all exercises for the student's own ingenuity are omitted. Certainly some of the long and cumbersome solutions could with advantage be replaced by such exercises. In adaptation to practical use there is a further deficiency in the second volume. The formulæ here, as in the first, are numbered with wearisome monotony from book cover to book cover, and the learner has no indication of which are deemed important, until he turns to the table at the end, and there unfortunately the numbering is entirely changed. In the first volume this fault is redeemed by the unusual excellence of the table. In the second, however, the summary of important integrals is decidedly inferior to corresponding summaries in our own text-books. But while these deficiencies on the practical side are to be regretted, the completeness of the general theory is a great satisfaction, especially in the integral calculus. The beginner is not only taught how to integrate, but what forms can, and what forms can not, be integrated. He learns definitely that any rational function whatever of the trigonometric ratios of an angle can be integrated; that $\int R(x, \sqrt[n]{f(x)}).dx$ can always be found in terms of elementary functions when $f(x)$ is linear, or, if $n = 2$, when $f(x)$ is a polynomial of the second degree; that when $f(x)$ is of a higher degree, integration is in general impossible without the introduction of new transcendental functions; and so on. Such simple statements as these are almost universally omitted in our treatises. The student has set before him classified lists of examples appropriate to the various methods, but as soon as he encounters new problems elsewhere, he is in painful uncertainty whether any of the methods will avail, and if so, which one. Integration therefore becomes mere experimentation. For this reason many a learner acquires a dislike for the integral calculus, which would certainly be obviated by so complete an exposition of the general theory as Kiepert's.

Still another trait which reveals itself in the work and one which is likewise characteristic of its nationality is thoroughness. This is well exemplified in the fifty pages of introduction with which the differential calculus opens. This is indeed one of its most striking and excellent divisions. The fundamental conceptions of calculus are admirably presented. The notion, and to some extent the properties also, of a function are discussed at considerable length. The book cannot be read in sublime unconsciousness that differentiation presupposes the continuity of a function. For the notion of continuity our space conceptions are first drawn upon, but the analytical criterion is also developed as fully as is advisable in an elementary text-book. Good illustrations of discontinuous functions are added. The notion of a limit is unfolded with a like completeness. The author, for instance, carefully guards against the common error of the learner that a variable is either always greater or always less than the limit which it approaches. Widely different concepts, such as velocity, infinite series, irrational numbers, circulating decimals, and the slope of a tangent to a curve, are presented under the form of a limit. The fundamental theorems of limits are established with entire strictness. Next follows the introduction of the infinitesimal and the infinite, and the foundations of an infinitesimal calculus are securely built upon the method of limits. The comparisons and illustrations of the different orders of infinitesimals are neat, as also is the proof that the sum of an infinite number of infinitesimals remains unaltered, if each infinitesimal is replaced by another differing from it by an infinitesimal of higher order. Finally the two chief methods of using infinitesimals—in quotients and in sums with an infinite number of terms—and the corresponding two-fold division of the calculus are indicated. Thus the introduction sets before the student clearly and conspicuously the fundamental conceptions of the science which too often are left only a nebulous mass without order or definition. Some glimpses are afforded him also of the wide applicabilty of the calculus, and his interest is excited even before he learns to differentiate.

A like thoroughness is shown, though not without a number of exceptions to be noted later, in the rigor of the demonstrations. Our own text-books are often altogether lacking in that strictness of reasoning which is demanded by modern mathematics. Salmon's text-books may here again be cited as examples. Among Germans, on the other hand, rigor of proof is insisted upon in comparatively early stages of mathematical training. Kiepert's treatise does not disappoint us in this regard. Great care is taken to note all exceptions and limitations to a theorem. The general thoroughness may be illustrated by reference to a few typical propositions which are most often loosely handled. In the proof that in the formation of the partial derivatives of $f(x, y)$ with respect to x and y, the order of differentiation is immaterial, the very question at issue is commonly begged by assuming that

$$\frac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)-f(x+\Delta x, y)+f(x, y)}{\Delta y\ \ \Delta x.}$$

has a limit independent of the order in which Δx and Δy approach zero. Kiepert's proof, however, faces this question, being based, like Todhunter's, upon Taylor's theorem. Again e^x is introduced in the customary manner as the limit of the series for $(1+\frac{1}{n})^n$ when n approaches ∞. Generally in the expansion of the expression the sum of an infinite number of infinitesimals is silently neglected, but Kiepert proves rigorously that the limit of the sum in this case is zero. The proof of this point is long and involved, and it may be questioned whether it is wise to force it upon a beginner, but in any event he should be told that without it the reasoning is incomplete. In a somewhat parallel case, where Kiepert establishes Taylor's theorem by the loose method of undetermined coefficients, he conscientiously calls attention to every assumption made. Another instance of exactness is found in the derivation of the series for $\tan^{-1}x$ and $\sin^{-1}x$ by integration of the series for their derivatives. This is accompanied with a proof that when a function can be developed into a series of ascending powers, its derivative also can be thus developed, and the second series is, term by term, the derivative of the first. How many mistakes in the applications of mathematics would be avoided, if those who made them had been trained to investigate the legitimacy of processes in which an infinite number of operations is used!

With this prevailing rigor a loose exposition of a number of points forms a glaring contrast. We certainly should not have expected to find the differential of the dependent variable defined as an infinitely small increment of the variable. Yet this definition is reiterated again and again. Surely, after the capital discussion of infinitesimals in the introduction, a line or two would have sufficed to show that the differential differs from such an increment by an infinitesimal of higher order and therefore can be employed in its place. Many of the common formulæ involving differentials, such as $(ds)^2 = (dr)^2 + r^2 (d\varphi)^2$ and $A = \int r^2 d\varphi$, are derived in the usual indifferent manner, although great care is taken to establish others rigorously. The old time definition of an algebraic function is given and with the usual inaccuracy; namely, in specifying the operations in which the variable is involved, there is no stipulation that the number of such operations should be finite. Thus the vital distinction between an algebraic and a transcendental function is entirely missed. It is but fair, however, to state that in a note the author gives the modern definition of an algebraic function as the root of an equation of the n^{th} degree whose coefficients are rational in the independent variable. An unfortunate omission is that of a proof of the formula $\frac{dx^n}{dx} = nx^{n-1}$ for incommensurable exponents. In the chapter on convergence of series the author announces that the terms should be summed in the order given, and then a few pages later, to prove the important theorem that a series containing both positive and negative numbers converges, if the sum of their moduli converges, he groups apart the positive terms and the negative terms without establishing the legitimacy of the change. This error is the more worthy of attention because it is prevalent in textbooks and occurs in the discussion of a subject, in which rigor should be strenuously insisted upon.

The chapter upon convergence and divergence of series is perhaps the least satisfactory in the entire work. It is apparently a recent insertion and the amalgamation is incomplete. Nothing hinges upon the chapter. Elsewhere the character of a series is uniformly investigated by the use of some one of the various forms of the remainder in Taylor's theorem. The prominence given to this remainder is indeed a marked feature of the first volume. It is even made the basis for a study of indeterminate forms, maxima and minima, points of inflection, etc. Now while its use leaves nothing to be desired in respect to rigor, it necessarily complicates the proofs, rendering them exceedingly difficult to the beginner. An equal degree of exactness could be obtained very simply by direct application of Cauchy's test for a power series; to wit, a series converges, if from and after some fixed term the ratio of each term to the preceding is numerically less than some fixed quantity less than unity. Certainly, considered in relation to the entire field of mathematics, it is this test for convergence rather than the other which should be emphasized as of fundamental importance. The true rôle of Taylor's remainder is not to determine the character of a series, but to estimate the magnitude of the error, when the series is broken off with a given term. Once, and once only, so far as I have noticed, has Kiepert used the remainder for this purpose. But the chief defect of the chapter is in the treatment of conditional and unconditional convergence. No one theorem pours such a flood of light upon the subject as Riemann's theorem that a conditionally convergent series can be made to assume any value by a rearrangement of the terms. But this theorem is omitted and in its place a somewhat complicated example of a conditionally convergent series is given. Furthermore, no emphasis is laid upon the fact that absolute and unconditional convergence are synonymous. It is practically degraded to the position of a corollary. If the proof be carefully examined, it will be found to be based upon a theorem for which one looks in vain, that a series is convergent, when, by proceeding far enough in the series, the sum of p consecutive terms ($p = 1, 2, \cdots \infty$) can be made as small as we choose, and conversely. I may add that there is no reference by name to the "commutative" and "associative" laws of operations, though both laws are implicitly used. The limitation of the latter law to convergent series is not observed, and in consequence an oscillating series is represented as having several limits instead of none.

In conclusion, a few minor points may be rapidly mentioned. The italicizing of the leading results, so common with German mathematicians, is to be unqualifiedly recommended, as also the frequency with which Kiepert attaches to the chief theorems the names of their inventors. This practice greatly helps to arouse the interest of the student in the works of the great masters. The frequent practical applications of the theorems, the determination of the value of π by the three methods of Leibnitz, Euler and Machin, the happy treatment in both volumes of simply-periodic series, the *Mittelwerthsätze* for definite integrals and their approximate computation by Simpson's rule, the extremely rapid method of computing A and B in $\dfrac{Ax + B}{(x-a)^2 + b^2}$, when a rational fraction is decomposed into partial fractions—these and other points are worthy of mention.

Taking the two volumes together, they form a valuable work of reference to the college teacher of calculus. To an unusual degree they give just what the student should know. The French sparkle is perhaps missing, but we are well satisfied with the German accuracy, thoughtfulness and thoroughness.

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TRIANGULAR NUMBERS. *

In a review † of Blater's Table of Quarter Squares, Mr. J. W. L. Glaisher, in 1889, referred to the convenience that would be afforded by a table of triangular numbers, and said that the only extensive published table of such numbers known to him was that by E. de Joncourt, published at the Hague in 1762, giving the values of $\frac{1}{2}n(n+1)$ from $n = 1$ to $n = 20{,}000$. Letting S_n represent the n^{th} triangular number, the sum of the n numbers up to and including n, or $\frac{1}{2}n(n+1)$, the application of a table of triangular numbers to facilitate multiplication is seen from the formula, $ab = S_{a-1} + S_b - S_{a-b-1}$. This formula shows the chief advantage claimed for the use of triangular numbers in preference to quarter squares—that the table need only extend as far as the highest number it is to be used to multiply. Thus, such a table need be only half as large as a table of quarter squares that may be used to multiply the same numbers by taking the difference between the quarter squares of the sum and difference of the factors. On the other hand the method of quarter squares requires but two instead of three tabular entries, and may be modified ‡ so as not to use an argument exceeding the larger factor; but in that case three tabular entries are required. A modification of the use of triangular numbers is also applicable to reduce the number of entries to two, but then we may need to use an argument greater than either factor.

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* Projet de Table de Triangulaires de 1 à 100,000, etc.; A. ARNAUDEAU (Paris: Gauthier-Villars et Fils, 1896).

† Reprinted in the *Journal of the Institute of Actuaries*, London, Jan. 1890.

‡ $ab = 2\left[\frac{a^2}{4} + \frac{b^2}{4} - \frac{(a-b)^2}{4}\right]$.