CONCERNING REGULAR TRIPLE SYSTEMS.

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§ 1.

Introduction. Definitions and notations.

A $k$-ad $[l_1, \ldots, l_k]$ $k$-id $\{l_1, \ldots, l_k\}$ is an arrangement of $k$ letters or elements $l_1, \ldots, l_k$ in which the order is not material. The letters are distinct.

A triple system $\Delta$ is an arrangement of $t$ letters in 3-adic triples in such a way that every 2-adic pair appears exactly once in some triple of the triple system. There is no question of order of the triples of the triple system. $t$ must have the form $t = 6m + 1$ or $t = 6m + 3$; $t$ denotes always a number of such (say) triple form.

A triple system $\Delta$ is invariant under a certain (largest) substitution group $G'$ on its $t$ elements; the $\Delta$ and the $G'$ belong each to the other; the $G'$ is a triple group.
In the definitive theory of triple systems and triple groups we must, among other things, determine whether a (any) particular group $G^t$ of triple degree $t$ is a triple group, and, if so, we must construct its various triple systems $A^t$.

A system $A^t$ is transitive if its group $G^t$ is transitive.

A transitive system $A^t$ is regular if its group $G^t$ contains a regular subgroup $H^t$ of order $t$ on the $t$ elements of the $A^t$; the group $H^t$ is a group of regularity of the system $A^t$; the system $A^t$ has with respect to the group $H^t$ the regular aspect $A^t/H^t$. A regular group $H^t$ is uniquely determined by the corresponding abstract group $H$.

Mr. Netto has exhibited, cyclic, or, as I say, cyclic-regular $A^t$ (1°) for $t = 6m + 1 = p$ (any prime) and (2°) for $t = 6m + 3 = 3p$ (any prime).

Mr. Heffter in the current number (vol. 49, part 1) of the Mathematische Annalen, has brought out clearly the "difference problems" underlying the problems of construction of the general cyclic-regular $A^t$ for $t = 6m + 1$, $6m + 3$. He then exhibits cyclic-regular $A^t$ (3°) for $t = 6m + 1 = 12k + 7 = 3p - 2$ (any prime of the form $p = 4k + 3$ with the primitive root 2) and (4°) (by a slight modification of Mr. Netto's (2°) system) for $t = 6m + 3 = 3p$ (any prime greater than 3).

Now a cyclic group $H^t$ is the simplest Abelian or commutative group $H$.

In a paper, Concerning Abelian-Regular Transitive Triple Systems, forwarded to the Annalen three weeks ago, I have analyzed (i.e., §1) the general Abelian-regular $A^t$, and have exhibited (i.e., (a) §2, (b) §§ 3, 4) Abelian-regular $A^t$, with respect to the abstract Abelian group $H$, where

(a) $t$ is any integer $t = 6m + 3$,

$h^t$ is any Abelian group of order $t$ having one invariant 3; or

(b) $t$ is any integer $t = 6m + 1$ in which every prime factor of the form $p = 6k + 5$ enters an even number of times;

$h^t$ is any Abelian group of order $t$ in whose invariant-character every such prime enters always with the exponent 1.

The systems (a, b) are sweeping generalizations of the systems (2°, 4°; 1°) of Messrs. Netto and Heffter.

My analysis of the general Abelian-regular system $A^t$ was so phrased as to admit of immediate application in the constructions (a, b). In this paper the general regular triple system $A^t$ is subjected to a similar analysis (§ 2), on the
§ 2.

Regular triple systems $\mathcal{A}_t \mid H_t$ and the corresponding sextette separations $\sigma_m, n \mid \tilde{H}_t$.

We denote the $t$ elements of the group $H_t$ and the $t$ elements of the regular system $\mathcal{A}_t \mid H_t$ without confusion by the same notation—$A_1, \ldots, A_t$. The substitution group $\tilde{H}_t$ is made up of the substitutions

\[ A_j = \left( A_1, \ldots, A_i, \ldots, A_t \right) \quad (j = 1, 2, \ldots, t). \]

The regular system $\mathcal{A}_t \mid H_t$ is invariant under the $t$ substitutions $A_j$ (1) of the $\tilde{H}_t$. Hence the $\mathcal{A}_t$ contains with the triple $[A_i, A_j, A_k]$ the $t$ triples (not necessarily all distinct)

\[ [A_i, A_j, A_k, A_j, A_k, A_j] \quad (j = 1, 2, \ldots, t). \]

Writing the triple or 3-ad $[A_i, A_j, A_k]$ having regard to the order of the letters as a 3-id $\{A_i, A_j, A_k\}$ we see that the corresponding sextette

\[ \sigma \{ A_i, A_j, A_k \} = \left\{ A_i A_j^{-1}, A_j A_k^{-1}, A_k A_i^{-1} \right\} \]

is an invariant and indeed a characteristic invariant for the 3-idic triples

\[ \{A_i, A_j, A_k, A_i, A_j, A_k\} \]

of this set (2).

Such a sextette $\sigma \{ A_i, A_j, A_k \}$ has the necessary and sufficient form

\[ \sigma \mid H_t = \left\{ B_i, B_j, B_k \right\} \left\{ B_i^{-1}, B_j^{-1}, B_k^{-1} \right\} \]

where\(^*\)

\[ B_i B_j B_k = I, \quad B_i \equiv I, \quad B_i \equiv B_j^{-1} \quad (i, j = 1, 2, 3), \]

\(^*\) The group $H_t$ is of odd order $t$. Every triple has three distinct letters. Two triples having two letters in common have also the third of each in common. The sextette $\sigma \mid H_t$ with $B_i B_j B_k = I$ belongs to the 3-idic triples

\[ (I, B_i^{-1}, B_j), \quad (B_i, I, B_k^{-1}), \quad (B_j, B_k, I). \]

From these remarks one draws conclusions (5, 6) of the text.
the $B$'s being elements and the $I$ the identity element of the group $H_t$. There are in all two types of sextettes $\sigma \mid H_t$:

$$
\begin{align*}
(1°) & \quad B_1B_2B_3 \text{ are distinct;} \\
(2°) & \quad B_1 = B_2 = B_3.
\end{align*}
$$

According as the sextette $\sigma \mid H_t$ is $\sigma_1 \mid H_t$ or $\sigma_2 \mid H_t$ of the type 1° or 2°, it contains six or two distinct elements and the corresponding set of triples (2) contains $t$ or $\frac{1}{2}t$ triples

and is indeed a (tactical) configuration$^*$ $Cf_1\left(\begin{array}{c} t \\ 3t \\
\end{array}\right)$ or $Cf_2\left(\begin{array}{c} t+1 \\ 3, \frac{1}{2}t \end{array}\right)$ regular with respect to the group $H_t$ of substitutions $A$, (1). The type 2° occurs only if $t$ has the form $t = 6m + 3$.

The system $A \mid H_t$ is the composition of $m_1$ configurations $Cf_1 \mid H_t$ of type 1° and $m_2$ configurations $Cf_2 \mid H_t$ of type 2°, with distinct triples. Here $tm_1 + \frac{1}{2}tm_2 = \frac{1}{3}t(t - 1), \therefore 3m_1 + m_2 = \frac{1}{3}(t - 1) = 3m$ or $3m + 1$. Hence we have

$$
(7) \quad t = \frac{6m + 1}{6m + 3}, \quad (m_1, m_2) = (m, 0), \quad (m - m', 1 + 3m')
$$

Corresponding to and characteristic of this configuration-separation $Cf_{m_1, m_2} \mid H_t$ of the system $A \mid H_t$ is a sextette-separation $\sigma_{m_1, m_2} \mid H_t$ of the $t - 1$ elements $A (A \equiv I)$ into $m_1$ sextettes $\sigma_1 \mid H_t$ and $m_2$ sextettes $\sigma_2 \mid H_t$ (in which repetitions of elements occur only within the individual sextettes $\sigma_2 \mid H_t$).

Conversely, with respect to any abstract group $H_t$ of order $t = 6m + 1, 6m + 3$ any such sextette-separation $\sigma_{m_1, m_2} \mid H_t$ of the $t - 1$ elements $A (A \equiv I)$ serves uniquely to define a regular triple system $A \mid H_t$.

We consider as essentially the same the six sextettes $\sigma_2 \mid H_t$ derived from the two

CONCERNING REGULAR TRIPLE SYSTEMS.

by cyclical permutation of their columns, since they arise from the six 3-idic triples corresponding to one 3-adic triple.

If in a sextette \( \sigma_1 \mid H_t \) (4) we interchange

\[
B_i, B_i^{-1} \quad (i = 1, 2, 3),
\]

the new say reciprocal sextette is \( \sigma_1 \mid H_t \) if and only if

\[
B_1^{-1} B_2^{-1} B_3^{-1} = I.
\]

This happens, for instance, always if \( H_t \) is an Abelian group. Two reciprocal sextettes of type 1° are essentially distinct.

§ 3

Explicit exhibition of a sextette-separation \( \sigma_{0,1+3m} \mid H_t = 3^k = 6m + 3, \)

where \( H_t \) is any group of order \( t = 3^k = 6m + 3 \) whose elements not the identity are all of period 3.

Corresponding to the \( 1 + 3m \) pairs of reciprocal elements \( A (A \in \subset I) \) of the group \( H_t \) we have the separation of those elements (of period 3) into \( 1 + 3m \) sextettes \( \sigma_j \mid H_t \) of type 2°.

For the (cyclid) Abelian \( H_t = 3^k \) generated by \( k \) generators each of period 3 this separation underlies the Abelian-regular \( A^* \mid H_t^{3^k} \) whose group is the linear group modulo 3. (Netto, Substitutionentheorie, pp. 224–234.)

§ 4

Explicit exhibition of \( 2^m \) sextette-separations \( \sigma_{m,1} \mid H_t = 6m + 3, \)

where \( H_t \) is any group of order \( t = 6m + 3 \) having a self-conjugate element \( A_0 \) of period 3 and a subgroup \( K_{2m+1} \) of order \( 2m+1 \) not containing \( A_0 \).

C denotes always an element of the subgroup \( K_{2m+1} \). The \( 2m \) elements \( C(C \notin I) \) separate uniquely into \( m \) pairs of reciprocal elements \( C, C^{-1} \). If the \( C_i(i = 1, \ldots, m) \) form a system of representatives of these \( m \) pairs, then so do the \( C_i^{-1}(i = 1, \ldots, m) \).

The group \( K_{2m+1} \) extends by \( A_0 \) to the group \( H_t = 6m + 3 \).

Since \( A_0 \) is a self-conjugate element in \( H_t \), it is commutative with every element \( C \). The elements \( A \) of \( H_t \) have the form \( C, CA_0, CA_0^2, \ldots, CA_0^m \).

The separation \( \sigma_{m,1} \mid H_t \) of the \( t - 1 = 6m + 2 \) elements \( A(C \notin I) \) of the \( H_t \) consists of the one sextette
CONCERNING REGULAR TRIPLE SYSTEMS. [Oct.,

and of the \(m\) sextettes \(\sigma^{(0)} | H_t\) depending upon a representative system \(C = C_1, \ldots, C_m\) of the \(m\) pairs of reciprocal elements \(C(C \mapsto I)\),

\[
\sigma^{(0)} | H_t = \begin{cases} 
CA_0, & CA_0^2, \quad C^{-1} \\
C^{-1}A_0^2, & C^{-1}A_0 C^2 \end{cases}
\]

or

\[
\sigma_{1}^{(0)} | H_t = \begin{cases} 
CA_0^2, & CA_0, \quad C^{-1} \\
C^{-1}A_0, & C^{-1}A_0^2, \quad C^2 \end{cases}
\]

The two sextettes \(\sigma^{(0)}, \sigma^{(1)}\) of (2) or (3) are essentially the same, while \(\sigma_{1}^{(0)} (2)\) and \(\sigma^{(0)} (3)\) are essentially distinct (reciprocals).

According to the choice of (2) or (3) for each pair \(C, C^{-1}\) we have in all \(2^m\) sextette-separations \(\sigma_{m,1} | H_t\) and so \(2^m\) regular \(A_t | H_t\) for every abstract group \(H_t\) of the character in question. In particular, since every Abelian group \(H_t\) with one invariant 3 is such a group \(H_t\), we have the Abelian-regular \(A_t | H_t\) (§ 1, α).

For still more general types of groups \(H_t\) of order \(t = 6m + 3\) we may by suitable modification of the preceding process exhibit a sextette-separation \(\sigma_{m,1} | H_t\). Thus, for example, for those \(H_t\) with the following properties:

1. the \(H_t\) has an element \(A_0\) of period 3,
2. the group \(I, A_0, A_0^3\) extends to the \(H_t\) by the identity and certain \(2m\) extenders \(C\) by pairs reciprocal,
3. a certain representative system \(C(i = 1, \ldots, m)\) of these \(m\) pairs is invariant under transformation by \(A_0\),
4. the system \(C, A_0, C, A_0^3 (i = 1, \ldots, m)\) is a representative system of these \(m\) pairs. This type of group \(H_t\) contains the type previously discussed and also, for instance, the group \(H_{21}\) generated by two generators \(A_0, C\) subject to the generational relations

\[
A_0^3 = I, \quad C^7 = I, \quad A_0 C = C^2 A_0.
\]

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