LIE'S DIFFERENTIAL EQUATIONS.*

SOPHUS LIE—*Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen.* Bearbeitet und herausgegeben von DR. GEORG SCHEFFERS. Leipsic, Teubner. 1891. 8vo, pp. xiv + 568.

These lectures constitute one of the courses of Lie's cycle which has been repeating itself at the University of Leipsic since 1886. The course serves the double purpose of an elementary introduction to the theory of continuous groups and an exposition of how that theory subordinates the various heterogeneous methods of integrating ordinary differential equations to one general method, the key to which is the notion of an infinitesimal transformation, first introduced by Lie at the inception of his theories. The lectures have been edited with the double object of both scientific and pedagogic usefulness. They are so designed that a fourth semester student of a German university is prepared to read them, and they should offer no difficulty to the American reader who is familiar with the processes of the infinitesimal calculus. The numerous problems and illustrative examples drawn from geometry and mechanics commend the book to the private student.

The book falls into five parts: I. The Notions—Infinitesimal Transformation and One Parameter Group of the Plane, chapters 1–4, pp. 1–85; II. Utility of the Notion of Infinitesimal Transformation in Differential Equations of the First Order in Two Variables, chapters 5–9, pp. 86–187; III. One Parameter Groups in Three Variables, chapters 10–13, pp. 187–286; IV. One Parameter Groups and Infinitesimal Transformations in n Variables, Application of these Notions to Differential Equations, chapters 14–20, pp. 286–472; V. Integration of Ordinary Differential Equations of the Second Order which Admit of a One Parameter Group, and Related Problems, chapters 21–25, pp. 473–566.

I. A point transformation is an operation by which a point is carried into the position of a point. Two equations of the form

$$x_1 = \varphi(x, y), \quad y_1 = \psi(x, y), \quad \frac{\partial(\varphi, \psi)}{\partial(x, y)} \neq 0, \qquad (1)$$

φ and ψ being regular analytic functions, are said to determine a point transformation of the plane into itself. If the equations (1) contain a parameter a, they define a family of

* An interesting account of this work, from a somewhat different point of view, was contributed by Professor E. Study to the *Zeitschrift für Mathematik und Physik*, vol. 38 (1893), pp. 185-192.—EDITORS.

∞^1 such transformations. Such a family of ∞^1 transformations constitutes a one parameter finite continuous group of transformations when the successive performance of any two transformations of the family is equivalent to a transformation belonging to the family, *i. e.*, if the ∞^1 transformations

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a) \qquad (2)$$

form a continuous group, the elimination of (x_1, y_1) between these equations and the two following

$$x_2 = \varphi(x_1, y_1, a_1), \quad y_2 = \psi(x_1, y_1\, a_1),$$

must give rise to

$$x_2 = \varphi(x, y, \alpha), \quad y_2 = \psi(x, y, \alpha),$$

where α is a function of a and a_1 alone.

Only those groups are studied which contain the inverse transformation of every transformation in them, *i. e.*, whose transformations can be arranged in pairs as inverse.* Accordingly, to every value of the parameter a there corresponds a value $\bar{a}$ such that

$$x = \varphi(x_1, y_1, \bar{a}), \quad y = \psi(x_1, y_1, \bar{a}).$$

Since the successive performance of a transformation and its inverse yields the identical transformation, there is some value of the parameter a, say a_0, which gives the identical transformation

$$x = \varphi(x, y, a_0), \quad y = \psi(x, y, a_0),$$

leaving all points at rest. A value of the parameter differing by an infinitesimal from that of the parameter of the identical transformation gives the transformation

$$x_1 = \varphi(x, y, a_0 + \delta a) = x + \xi(x, y)\, \delta t + \cdots,$$
$$y_1 = \psi(x, y, a_0 + \delta a) = y + \eta(x, y)\, \delta t + \cdots,$$

by which the point (x, y) is transformed into a point $(x + \delta x, y + \delta y)$ infinitely near, where

$$\delta x = \xi(x, y)\, \delta t, \quad \delta y = \eta(x, y)\, \delta t,$$

to terms of the second order. Such a transformation is called an infinitesimal transformation.

Consider now several examples of ordinary differential equations that are readily integrable. The differential

* This limitation is really no restriction. It is only formal since Lie has devised means for deriving the defining equations of any finite continuous group from those of a group whose transformations are inverse in pairs. See Theorie der Transformationsgruppen, vol. 3, theorem 26.

equation $y' = f(x + ky)$ assigns the same direction y' to every point of the straight line

$$x + ky = constant. \tag{3}$$

Hence all the integral curves of this equatiôn may be derived from any one of them by the translation of that one along the parallel straight lines (3). These translations however form a one-parameter group defined by the equations

$$x_1 = x - ka, \quad y_1 = y + a.$$

By the transformations of this group the integral curves of the above differential equation are changed one into another.

The homogeneous equation

$$y' = f\left(\frac{y}{x}\right)$$

gives to every point of the straight line $y = mx$ the same direction; hence the integral curves of the homogeneous differential equation can be found from any one of them by proportionately increasing or diminishing the latter from or toward the origin. These proportionate increasings and diminishings from the origin out constitute a group of operations, namely the one parameter group of so-called similitudinous transformations, $x_1 = ax$, $y_1 = ay$. Here again the family of integral curves as a whole remains invariant, the curves of the family being interchanged among themselves when all the transformations of the one parameter group are performed.

As a third example the integrable form

$$xy' - y - (x + yy')f(x^2 + y^2) = 0$$

assigns to the points of a circle with centre at the origin directions which make the same constant angle with the circle, as is shown by the equivalent form

$$\left(y' - \frac{y}{x}\right) / \left(1 + y'\frac{y}{x}\right) = f(x^2 + y^2),$$

where the left hand member represents the cotangent of the angle in question. Hence all the integral curves cut the circle under the same angle and accordingly they may all be found by rotating any one of them about the origin. All these rotations about the origin make up the one parameter group, $x_1 = x \cos a - y \sin a$, $y_1 = x \sin a + y \cos a$.

It will also be observed that in any one of these illustrations an infinitesimal transformation of the group considered changes integral curve into an integral curve. In each case the differential equation or the family of integral curves is said to admit of the one parameter group of transformations. These examples show that when the integration of a differential equation can be effected there can be given a group of point transformations which transform the points of an integral curve into those of an integral curve. This fact suggests the high probability that the knowledge of such a group of transformations may be used to simplify and indeed methodically to systematize the processes of integration. It is largely the business of the sequel of the book before us to reduce this probability to a certainty and thereby to develop a general method of integration of ordinary differential equations on the connection between differential equation and group of transformations pointed out in the above simple examples. This general method is the outgrowth of Lie's discoveries made in the years 1869 to '74.

The intimate relationship between the notions one parameter group and infinitesimal transformation is of prime importance. In fact from Lie's theorem that a one parameter group contains but one infinitesimal transformation and conversely that an infinitesimal transformation carried out successively generates but one one parameter group,* the two notions may be regarded as coextensive.

Two infinitesimal transformations whose ξ and η differ by the same constant factor are said to be dependent; they are in no way essentially different, since δt is arbitrary. The finite equations of the group generated by the infinitesimal transformation are found by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt,$$

with the initial conditions

$$x_1 = x, \quad y_1 = y, \quad t = 0,$$

in the form

$$\Omega(x_1, y_1) = \Omega(x, y), \quad W(x_1, y_1) - t = W(x, y), \qquad (4)$$

or solved with regard to x_1, y_1 and developed in powers of t,

$$x_1 = x + \xi(x, y)\, t + (\xi \xi_x + \eta\, \xi_y) \frac{t^2}{1 \cdot 2} + \cdots,$$

* It is to be noted that when the term group is used in this review a one-parameter group of transformations inverse in pairs is meant. Similarly transformation in any geometrical connection is short for point transformation.

$$y_1 = y + \eta(x, y)\, t + (\xi\eta_x + \eta\,\eta_y)\frac{t^2}{1\cdot 2} + \cdots.$$

The infinitesimal transformation appears again by putting δt for t.

If we remark that a change of variables does not affect the group property, by the substitution

$$\Omega(x, y) = \bar{x}, \quad W(x, y) = \bar{y},$$

the above equations (4) assume the form

$$\bar{x}_1 = \bar{x}, \quad \bar{y}_1 = \bar{y} + t\,;$$

hence every one parameter group can be brought to the form of a group of translations by a proper change of variables. This form Lie calls the canonical form of the group and the reducing variables its canonical variables.

Lie adopts

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y} \equiv \xi p + \eta q$$

as the symbol of an infinitesimal transformation. This choice of a symbol, which, by the theorem quoted above, may represent the infinitesimal transformation or the one-parameter group indifferently, is a peculiarly happy one, because of several of its properties: 1° $Uf\delta t$ is the increment which an arbitrary function $f(x, y)$ receives by the transformation; 2° an arbitrary function $\varphi(x_1, y_1)$ is given by the series

$$\varphi(x_1\, y_1) = \varphi(x, y) + \frac{t}{1}U\varphi(x, y) + \frac{t^2}{1\;2}UU\varphi(x, y) + \cdots;$$

3° the form of Uf is unchanged when new variables, functions of the old, are introduced; 4° the commutator* of any two Uf's is a symbol of the same form, namely,

$$\begin{aligned}(U_1U_2)f &\equiv U_1(U_2f) - U_2(U_1f)\\ &\equiv (U_1\xi_2 - U_2\xi_1)p + (U_1\eta_2 - U_2\eta_1)q\,;\end{aligned}$$

5° the symbol of an infinitesimal transformation of an r parameter group is expressible linearly with constant coefficients in symbols of the above form; 6° it is a convenient

* This may perhaps be taken as an English equivalent of *Klammerausdruck;* it is suggested by the fact that the vanishing of the expression is the condition that the two operations be commutative. A direct translation of the term would be very cumbersome.

representative of a group when the group becomes the object of another group. All necessary computations are made with this symbol. This operator U is to the theory of continuous groups what the operation of differentiation is to the calculus.

Since the symbol Uf preserves its form when new variables are introduced every one parameter group may be changed into every other. If $x'(x, y)$ and $y'(x, y)$ are the new variables,

$$U'f = Ux' \frac{\partial f}{\partial x'} + Uy' \frac{\partial f}{\partial y'};$$

hence the solutions of $Ux' = 0$ and $Uy' = 1$ give the canonical variables of the group Uf; these canonical variables are accordingly determined by an integration followed by a quadrature.

One of the first questions to arise in studying a family or group of transformations is—What functions, equations and geometrical configurations are invariant by the transformations? If the transformations do not form a continuous group they may or may not have invariants;* if they form a continuous group the group must have invariants. At this point Lie establishes the beautiful theorem—In order that a function, equation or curve admit of all the finite transformations of a one parameter group it is necessary and sufficient that it admit of the infinitesimal transformation of the group.

The invariant functions are the solutions of the partial differential equation $Uf = 0$; an equation $\omega(x, y) = 0$ is invariant if $U\omega = 0$, by virtue of $\omega = 0$; an invariant curve is either 1° a path curve,† *i. e.*, the locus of all the positions which a point takes when subjected to all the transformations of the group, or 2° a curve all of whose points remain absolutely at rest by every transformation of the group. The first are given by equating an arbitrary invariant function to a constant; the second by the common solutions of the equations $\xi = 0$, $\eta = 0$. The first part concludes with a brief study of the projective, conformal and area-preserving point transformations of the plane relative to the points and properties of the preceding sections just enumerated.

* The ∞^1 transformations $x_1 = x + 1$, $y_1 = y + t$ do not form a group; $\tan \pi x$ is an invariant function but not every function of $\tan \pi x$; the family of curves $\tan \pi x = k$ is invariant. The ∞^1 transformations $x_1 = xt$, $y_1 = y + t - 1$ have no invariant function. The latter family contains the identical transformation and an infinitesimal transformation. There is a system of ∞^2 straight lines invariant by the family.

† Bahncurve; trajectoire.

II. In order to successfully apply the theory of groups to differential equations admitting of known infinitesimal transformations there remains still a question to be met— When does a family of ∞^1 curves admit of the transformations of a group? Lie gives the answer in the following theorem: A family of curves, $\omega(x, y) =$ *constant*, admits of the group Uf in the case when $U\omega = \Omega(\omega)$ and in no other. If each curve of the family is invariant, $\Omega = 0$; if the curves of the family are interchanged among themselves, $\Omega \neq 0$; and in the latter case Ω may be taken equal to unity. The preceding theorem combined with the fact that the integral curves, $\omega(x, y) =$ *constant*, of an ordinary differential equation $X(x, y)dy = Y(x, y)dx$, may be defined by the solution ω of the associated homogeneous linear partial differential equation

$$Af \equiv X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial x} = 0,$$

leads to the following theorem first published by Lie in 1874 and establishing the fundamental connection between the integrating factor of Euler and the infinitesimal transformation of Lie: If the differential equation $Xdy - Ydx = 0$ admits of the known infinitesimal transformation $Uf = \xi p + \eta q$, where $X\eta - Y\xi \neq 0$, then

$$\frac{1}{X\eta - Y\xi}$$

is a Eulerian multiplier of the equation and the equation of the integral curves is

$$\int \frac{Xdy - Ydx}{X\eta - Y\xi} = \text{constant.}$$

If $X\eta - Y\xi = 0$, each integral curve is invariant by itself and the transformation Uf is of no avail; Uf in this case is said to be *trivial* with regard to the given equation.

For the practical application of this theorem there is necessary a criterion that a given differential equation admit of an infinitesimal transformation Uf. Lie finds this criterion to lie in the demand that the commutator $(UA)f$ shall be identically equal to $\lambda(x, y) \cdot Af$.

Conversely, if we are given a multiplier M of a differential equation, we shall have an infinitesimal transformation of which the equation admits, by determining ξ and η from the condition $\dfrac{1}{X\eta - Y\xi} = M$, an indeterminate equation which shows that every differential equation of the first order in two variables admits of an infinite number of infinitesimal transformations. A similar theorem does not hold for differential equations of a higher order.

If Uf is reduced to its canonical form the equation admitting of it assumes the immediately integrable form $dy' - F(x')\,dx' = 0$. If the equation admits of U_1f and U_2f then $(X\eta_1 - Y\xi_1) / (X\eta_2 - Y\xi_2)$ is either an integral of the equation or a constant; finally if the equation admits of the non-trivial transformation Uf it admits of $\Omega(\omega)\,Uf + \varphi(x, y)\,Af$.

On the other hand if we start from the finite equations of the group Uf, we can determine by differentiation and elimination, all ordinary differential equations of the first order in x and y which admit of the group generated by Uf. In this way Lie finds all the known cases of integrability. For example for the homogeneous equation Uf is $xp + yq$ and for the general linear equation Uf is $e^{\int \phi(x)dx}\dfrac{\partial f}{\partial y}$; the finite equations of these groups are $x_1 = ax$, $y_1 = ay$; $x_1 = x$, $y_1 = y + ae^{\int \phi(x)dx}$, respectively.

The second part concludes with a chapter on geometrical applications of the theory of integration developed in this part. It includes, among other things, Lie's geometrical interpretation of Euler's multiplier and a number of the theorems relative to curves on surfaces that appeared in Lie's earlier geometrical work.

III. The first chapters of the third part repeat the theorems of the first part for groups in three variables. It has been remarked that the applications were made in the preceding articles to the explicit form $Xdy - Ydx = 0$; by introducing the notion *extended* group Lie makes possible the direct application of these theorems to the general form $\Omega(x, y, y') = 0$. The transformation in the three variables x, y, y'

$$x_1 = \varphi(x, y),\ y_1 = \psi(x, y),\ y_1' = \frac{dy_1}{dx_1} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = \chi(x, y, y'),$$

Lie defines as the first extension of the point transformation $x_1 = \varphi$, $y_1 = \psi$; it is also called the once extended point transformation. If the transformation belongs to a G_1 the extended transformation belongs to a G_1 which is the extended group of the first G_1. To the infinitesimal transformation $Uf = \xi p + \eta q$ corresponds the once extended infinitesimal transformation

$$U'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \left(\eta_x + y'(\eta_y - \xi_x) - y'^2\xi_y\right)\frac{\partial f}{\partial y'}$$

$$\equiv \xi(x, y) \frac{\partial f}{\partial y} + \eta(x, y) \frac{\partial f}{\partial y} + \eta'(x, y, y') \frac{\partial f}{\partial y'} = Uf + \eta' \frac{\partial f}{\partial y'} \cdot$$

A differential equation $\Omega(x, y, y') = 0$ may now be regarded as an algebraic equation in the three variables x, y, y' and the group $U'f$ as a G_1 in the same three variables and we have the criterion: The differential equation $\Omega(x, y, y') = 0$ admits of the group $U'f$, and, therefore, Uf, in the case when $U'\Omega = 0$, either identically, or by virtue of $\Omega = 0$, and in no other case. Conversely all differential equations $\Omega = 0$ which admit of a given group Uf are found by solving the problem of finding all surfaces in x, y, y' which admit of $U'f$. The latter problem is solved in the same way as the problem already handled in the plane. Any two independent solutions u and v of the linear partial differential equation $U'f = 0$ are two invariants and every invariant of the group is a function of these two. The ∞^2 pathcurves are given by the equations $u =$ *const.* and $v =$ *const.* There are two kinds of invariant surfaces: 1° those generated by ∞^1 path curves and represented by a single arbitrary equation in u and v; 2° those made up of invariant points. Lie calls an invariant of $U'f$ a differential invariant of the first order of the original group Uf; hence his theorem that every differential equation $\Omega(x, y, y') = 0$ which admits of Uf is found by equating a differential invariant to zero.

IV. Extending the preceding notions to the case of n variables, the necessary and sufficient condition that

$$Af = \sum_1^n \alpha_i \frac{\partial f}{\partial x_i} = 0 \text{ admit of } Uf = \sum_1^n \xi_i \frac{\partial f}{\partial x_i} \text{ is that}$$

$$(UA) \equiv \lambda(x_1, \cdots, x_n) Af.$$

By means of the expression

$$((U_iU_j)A) + ((U_jA)_iU) + ((AU_i)U_j) \equiv 0,$$

which he calls the identity of Jacobi, Lie finds that if $Af = 0$ admits of U_if and U_jf it also admits of $(U_iU_j)f$. If $Af = 0$ admits of $n - 1$ known infinitesimal transformations

$$U_jf = \sum_1^n {}_k\xi_{jk} \frac{\partial f}{\partial x_k},$$

which are linearly independent of each other and Af, then a multiplier of Af is given by

$$\frac{1}{M} = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n-1,1} & \xi_{n-1,2} & \cdots & \xi_{n-1,n} \end{vmatrix}.$$

Every differential equation of an order higher than the first does not necessarily admit of an infinitesimal transformation. The most general differential equation of the second order which admits of the infinitesimal transformation or G_1, Uf, is found by equating to zero the most general invariant of the twice extended group

$$U''f = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''},$$

where

$$\eta' = \frac{d\eta}{dx} - y'\frac{d\xi}{dx}, \quad \eta'' = \frac{d\eta'}{dx} - y''\frac{d\xi}{dx}.$$

The most general invariant of $U''f$, which by definition is also a differential invariant of the second order of Uf, is an arbitrary function of u, v, and $w = dv : du$, where u is an invariant of Uf, v a first order differential invariant of Uf to be found by a quadrature if u is known. In a similar manner all differential equations of a higher order which admit of Uf are found by differentiation and quadrature if a zero order differential invariant of Uf is known. The criterion that $\Omega(x, y, y', y'') = 0$ admit of Uf is that $U''\Omega = 0$, either identically, or by virtue of $\Omega = 0$. If $\Omega = 0$ admits of a known Uf its integration demands the integration of two differential equations of the first order and two quadratures. If $\Omega(x, y, y', y'') = 0$ be taken in the solved form $y'' = \omega(x, y, y')$, then the latter admits of Uf, if the equivalent linear partial differential equation

$$Af = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(x, y, y')\frac{\partial f}{\partial y'} = 0,$$

admits of $U'f$. In the latter event $Af = 0$ and $Uf = 0$ form a so-called complete system, that is they have a solution in common, which according to a theorem of du Bois-Reymond may be found by the integration of an ordinary differential equation of the first order in x, y, and a second solution is given by a quadrature.

A study of differential equations of the second order that admit of several infinitesimal transformations leads to the notion *group of infinitesimal transformations*. r independent infinitesimal transformations $U_1f, \cdots, U_rf$ form an r parameter group when the commutators $(U_iU_k) = \sum_1^r {}_s c_{iks} U_s f$; the constants c_{iks} determine the structure of the group. If $\rho < r$ of the r infinitesimal transformations $U_1f, \cdots, U_\rho f$ form a group, the latter is called a ρ parameter subgroup, which, in particular, is an invariant subgroup if the commutators of $U_1f, \cdots, U_\rho f$ with all the $U_1f, \cdots, U_rf$ belong to the subgroup. The commutators (U_iU_k) above form an invariant subgroup, the so-called first derived group of the original. Every infinitesimal transformation of the group belongs to at least one two-parameter subgroup which can always be found by algebraic process.

The aggregate of all the infinitesimal transformations of which a differential equation of the second order can admit forms at most an eight-parameter group; the equivalent linear partial differential equation in (x, y, y') admits of the group of the extended infinitesimal transformations; the latter group is determined by the theorem that the extension of a commutator is identical with the commutator of the extended transformations. Hence we see from the preceding that if an equation of the second order admits of two or more independent infinitesimal transformations it admits of at least one two-parameter group. This shows the necessity of reducing the two-parameter groups of the plane to canonical forms. The integration of the equation in these circumstances Lie effects in the two following ways:

1°. By reducing the two-parameter groups of the xy-plane to their four canonical forms: p, q; q, xq; $q, xp+yq$; q, yq; the first three reductions demand at most two quadratures, the last the integration of an ordinary equation of first order; and introducing the reducing or canonical variables in the given equation of the second order it assumes respectively one of the following forms: $y'' - \varphi(y') = 0$; $y'' - \varphi(x) = 0$; $xy'' - \varphi(y') = 0$; $y'' - y'\varphi(x) = 0$; whose integration demand at most two quadratures.

2°. By using the equivalent partial differential equation. The integration of a linear partial differential equation $Af = 0$ in the three variables x, y, z that admits of two different infinitesimal transformations involves either two quadratures or the integration of a differential equation of the first order in x, y. From this it follows that the integration of an ordinary differential equation of the second order admitting of a two-parameter group can be effected by two independent or dependent quadratures at most.

V. The fifth part is chiefly concerned with differential equations admitting of three-parameter groups. By the aid of the notions subgroup, derived group and invariant subgroup referred to in the review of the fourth part, all three-parameter groups of the plane are classified. If the three infinitesimal transformations are U_1f, U_2f, U_3f, the commutators $(U_1U_2)f$, $(U_1U_3)f$, $(U_2U_3)f$ form the first derived group of the original one. There are six different types of structure of three-parameter groups in the plane, one, two, two, one, respectively, according as this first derived group is three, two, one, or no parameter. Further division according to the nature of the path curves gives, all told, thirteen different types of three-parameter groups. These results are interpreted geometrically by representing every infinitesimal transformation of the family of transformations $c_1U_1f + c_2U_2f + c_3U_3f$ by a point in the plane whose homogeneous coordinates are c_1, c_2, c_3. The commutator then expresses the relation between pole and polar with regard to a fixed conic.

The problem of reducing a given three-parameter group to its type in the above scheme of thirteen types falls into two problems: 1° to determine the type; 2° to determine the canonical variables. In Lie's terminology the first problem is to *norm* the given group; the norming of a group in this case is the forming of the first derived group and observing to what type of structure it belongs. The determination of the canonical variables involves only algebraic operations, or, at most, quadratures, except in the case of two unfavorable types whose reduction demands the integration of a differential equation of the first order; but these unfavorable cases do not appear in the integration theory of equations of the second order since there are no such equations admitting of the groups in question. The problem of integrating a differential equation of the second order admitting of a three-parameter group can be solved if the two following admit of solution: 1° to reduce a three-parameter group to its canonical form; 2° to integrate a differential equation of the second order which admits of one of the eleven canonical forms. By the preceding the solution of the first involves only possible operations; the eight different types of equations of the second order admitting of one or more of the eleven canonical forms are immediately integrable; hence the integration of an equation of the second order admitting of a three-parameter group is effected by possible operations involving no more than quadratures in the most unfavorable cases. In most cases the introduction of the canonical variables can be avoided and the integration performed by operations purely algebraic, if the integration problem be referred to that of the equivalent linear partial differential equation which admits of necessity of the extended group $U_1'f$, $U_2'f$, $U_3'f$; here again the most unfavorable case exacts no more than a quadrature.

The concluding chapter shows how the application of the methods of the book may be made to differential equations of the third order in two variables having known infinitesimal transformations and to partial differential equations of the first order in four variables admitting of three-parameter groups. If the first derived group of the latter has fewer than three parameters the integration is affected by three quadratures, the first two or last two of which are independent; if the first derived group has three parameters the integration of a Riccati equation is demanded.

A paragraph relative to the meaning and importance of the theories in exposition here for the general theory of differential equations, calling attention among other points to analogies with Galois' theory of algebraic equations, concludes this, the introductory volume of Lie's published works.

EDGAR ODELL LOVETT.

PRINCETON, N. J.,
5 *November*, 1897.

SHORTER NOTICE.

Famous Problems of Elementary Geometry. An authorized translation of F. KLEIN'S *Vorträge über ausgewählte Fragen der Elementargeometrie*, by WOOSTER WOODRUFF BEMAN and DAVID EUGENE SMITH. Boston and London, Ginn and Company, 1897. 12mo, pp. ix+80.

Whatever opinion one may hold privately as to the desirability of translations in general, the appearance of a readable English version of Professor Klein's pamphlet* can excite no feeling other than that of satisfaction. This lucid exposition of the historical and actual significance of the three great problems of Greek geometry is addressed to all interested in the teaching of elementary mathematics,

*Leipzig, B. G. Teubner, 1895.