sideration is however postponed until Section VII. In Section V the transformation by reciprocal polars is first taken up, then the general correlation is considered and finally the group consisting of all projective transformations and all correlations in space of three dimensions is treated by means of line coördinates. In Section VI the general subject of contact transformations is introduced and in Section VII a special contact transformation due to Lie is discussed. Finally in Section VIII transformations in space of more than three dimensions are considered, considerable attention being paid to the relation between space of five dimensions and line geometry on the one hand and sphere geometry on the other hand.

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GOURSAT'S PARTIAL DIFFERENTIAL EQUATIONS.

Leçons sur l'intégration des équations aux dérivées partielles du second ordre a deux variables indépendantes. Par E. GOURSAT, Professeur de calcul différentiel et intégral à l'Université de Paris.

Tome I. *Problème de Cauchy.—Caractéristiques.—Intégrales intermédiaires.* Paris, A. Hermann, 1896. 8vo, viii + 226 pp.

Tome II. *La méthode de Laplace.—Les systèmes en involution.—La méthode de M. Darboux.—Les équations de la première classe.—Transformations des équations du second ordre.—Généralisations diverses.* Paris, A. Hermann, 1898. 8vo, i + 344 pp.

THESE two volumes constitute a fitting sequel to the author's volume* and that of Mansion† on partial differential

* Goursat : "Leçons sur l'intégration des équations aux dérivées partielles du premier ordre, rédigées par C. Bourlet." Paris, A. Hermann, 1898. 8vo, 354 pp. The subjects studied and their order of development in this volume may be of interest here in connection with the above volumes of the series. They are as follows : Théorèmes généraux sur l'existence des intégrales.—Équations linéaires. Systèmes complets.—Équations linéaires aux différentielles totales.—Équations de forme quelconque. Généralités. Méthode de Lagrange et Charpit.—Méthode de Cauchy. Caractéristiques.—Définition des expressions (ϕ, ψ) et $[\phi, \psi]$. Première méthode de Jacobi.—Méthode de Jacobi et Mayer.—Méthode de Lie.—Étude géométrique des équations a trois variables. Courbes intégrales. Solutions

equations of the first order. Goursat's investigations in this field give him an eminent right to write an exposition of the present state of our knowledge in the domain of partial differential equations of the second order in two independent variables. The theory of partial differential equations has been the subject of such extensive investigations and the task of collecting and coördinating these researches into a systematic treatise is such a difficult one that it is natural to inquire into the general plan and spirit of Goursat's work. This inquiry is anticipated in the preface of the first volume.

Goursat observes that, chiefly under the influence of the writings of Fourier, Cauchy, Riemann, and their disciples, the search for the general integral of a partial differential equation, which is impracticable in most cases, has been gradually replaced by the study of the properties of particular integrals satisfying certain limiting conditions together with the search for the latter integrals. The limiting conditions can be varied in an infinite number of ways, but the majority of problems treated so far can be referred to two distinct types. Cauchy has shown that, in general, an integral is determined if we can give a curve situated upon this integral surface and the tangent plane at every point of the curve, provided that we suppose this integral to be represented by a development in integral series; the search for this particular integral is known as *Cauchy's problem.* On the other hand, in the realm of the real variable, an integral can be defined by the continuous sequence of values which it takes along a closed contour, the integral and its derivatives remaining continuous within the contour; this is the celebrated *problem of Dirichlet* for the equation of Laplace; recent investigations, notably those of Picard, make a similar treatment of more general equations possible.

The first volume is occupied exclusively with the problem of Cauchy. The second volume is devoted to the methods of Laplace and Darboux. It was the author's purpose in the first volume to conclude the second by a chapter on partial quadratures; that this is not realized is a disappointment to the reader, notwithstanding the fact that

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singulières. Théorie générale de Lie.—Transformations de contact.—Théorie des groupes. Méthode générale d'intégration.

Autorisirte deutsche Ausgabe von Maser. Mit einem Begleitwort von Sophus Lie. Leipz g, Teubner, 1893. 8vo, xiii + 416 pp.

† Mansion: "Théorie des équations aux dérivées partielles du premier ordre." *Mémoires de l'Academie de Belgique*, vol. 25 (1875). Vom Verfasser durchgesehene und vermehrte deutsche Ausgabe. Mit Anhängen von S. von Kowalevsky, Imschenetsky, und Darboux. Herausgegeben von H. Maser. Berlin, Springer, 1892. 8vo, xxii + 489 pp.

greater unity of treatment results from the introduction of the chapter on generalizations.

The exposition is excellent, exhausting the literature on the subject. The choice, development, and arrangement of the material leaves little to be desired, but there are those who may question whether the distribution of credit for the long and leading lines of the theory has been made with the same unerring judgment; in particular the extravagant claims for Ampère fail to do justice either to Lie as the inventor of contact transformations and of the idea of manifoldnesses as applied to integration problems* or to Goursat's own researches in the general theory of characteristics.

It is proposed here to follow the development with as much detail as the scope of a review will permit; the numbers attached to the sections correspond to the successive chapters of the work.

1. Goursat† introduces the general notions and the problem of the first volume by means of a particular class of equations,‡ namely, those of the surfaces generated by the curves of a complex or enveloped by the surfaces of a complex. In geometrical parlance a general integral of a linear partial differential equation of the first order is obtained by taking a simply infinite system of curves of a congruence, while the general integral of a non-linear equation of the first order is formed by the envelope of a simply infinite system of surfaces depending on two parameters, an arbitrary relation having been established between the two parameters.§ An obvious generalization consists in in-

* See Lie: "Geometrie der Berührungstransformationen," p. 520.

† The matter of this chapter is drawn largely from the following memoirs:

Darboux: "Memoire sur les solutions singulières des équations aux dérivées partielles du premier ordre," IV^e partie, *Mémoires présentés par divers savants*, etc., vol. 27.

Lie: "Ueber Complexe, inbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differentialgleichungen," *Mathematische Annalen*, vol. 5.

Klein: "Ueber gewisse in der Liniengeometrie auftretende Differentialgleichungen," *Mathematische Annalen*, vol. 5.

‡ Unless otherwise specified, when the term "equation" is used in the succeeding paragraphs of this review a partial differential equation of the second order is meant.

§ The distinction made here between the general integrals of the linear and non-linear forms is, as Goursat notes, not absolutely essential, since the general integral of a linear first order equation can be represented, and in fact in an infinite number of ways, by formulæ of the form

$$V[x, y, z, \phi(a)] = 0, \quad \frac{\partial V}{\partial a} + \frac{\partial V}{\partial \phi(a)} \phi'(a) = 0.$$

creasing the number of parameters; for example, if we consider curves or surfaces depending on three parameters we are led to second order equations whose theory presents striking analogies to that of equations of the first order. Thus, all the surfaces of a complex of curves whose three parameters are connected by two relations satisfy the same equation

$$L^2r + 2LMs + M^2t + N = 0, \qquad (1)$$

L, M, N being functions of x, y, z, p, q alone. The integral surfaces of this equation are characterized by the fact that through every point of one of these surfaces there passes one curve of the complex which curve osculates the surface. These curves of the surface which are tangent at all their points to the curves of the complex which pass through these points are called *lines of osculation*. It may happen that there is a first order equation

$$F(x, y, z, p, q) = 0 \qquad (2)$$

whose integral surfaces are integrals of (1), but not surfaces of the complex; in that case the equation (2) is said to be a singular integral of the first order of (1).

Similarly, starting from a complex of surfaces whose parameters satisfy two conditions we arrive at the particular form

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0, \qquad (3)$$

H, K, L, M, N containing only x, y, z, p, q.

The principles used in the derivation and discussion of the equations (1) and (3) are aptly illustrated by five geometrical examples. In connection with the notions relative to the contact and osculation of surfaces, Lie's remarkable transformation which changes straight lines into spheres is introduced. The utility of Lie's contact transformations for the general theory is indicated by the theorem that any equation belonging to the class (1) or (3) can be brought by a suitable contact transformation to the form

$$s^2 - rt = 0, \qquad (4)$$

the equation of developable surfaces.

Goursat adopts as the definition of the general integral of an equation of the second order

$$F(x, y, z, p, q, r, s, t) = 0, \qquad (5)$$

the definition which Darboux* has deduced from the works of Cauchy: An integral is general, if we can dispose of the arbitraries which figure in it, be they functions or constants unlimited in number, in such a manner as to find the solutions whose existence the theorems of Cauchy make known, *i. e.*, in such a manner as to attribute to the unknown function and one of its first derivatives values succeeding according to any continuous law given in advance for all the points of a curve. Cauchy's theorem does not demonstrate the existence of singular integrals; the general integral gives all the integrals except the singular integrals. It is the object of the present work to seek those integrals which are made known by the theorem of Cauchy; in every case where the solution can be referred to the integration of one or more systems of ordinary equations the question is considered solved.

Goursat points out that the frequent statement that the general integral of an equation in two independent variables depends on two arbitrary functions of one variable must not be taken literally, and illustrates the point by the equation

$$r - q = 0$$

whose general integral

$$z = \int_{-\infty}^{+\infty} e^{-u^2} \varphi (x + 2u \sqrt{y}) \, du,$$

due to Ampère, contains but one arbitrary function φ.

The chapter is concluded by a long section showing how the method of integration of the equations considered at the beginning of the chapter may be regarded in a certain sense as a generalization of the method of the variation of arbitrary constants.

The final sentence of this chapter † and a remark in the preface ‡ associate Ampère's name with the contact transformations of Lie in a way that may lead to misinterpretation. It must not be inferred that Ampère anticipated Lie's theory. Transformations applied to given differential equations appeared as early as Euler, Lagrange, and Legendre; but none of these mathematicians studied these

*Darboux: "Leçons sur la théorie générale des surfaces," vol. 2, p. 98.

† See page 38 and the reference to the 18th volume of the *Journal de l'École Polytechnique.*

‡ See page vii of the preface.

transformations in themselves or established general propositions concerning the particular transformations, much less concerning general categories of transformations. Ampère applied more general transformations than those of Euler and Lagrange to partial differential equations, but to Lie is due both the credit of inventing the idea of a contact transformation and that of establishing the theory of such transformations in an independent existence.

2. The second chapter * takes up the problem of Cauchy for the equation of Monge and Ampère. Using Ampère's notation a linear equation in r, s, t, $rt - s^2$ is written in the form (3). The notion *characteristic manifoldness* is at the basis of the theories of Monge and Ampère for the integration of equations of this form. Goursat introduces the notion by means of the problem of finding an integral surface of the equation (3) passing through a given curve C, and tangent all along C to a given developable D passing through this curve. The one dimensional manifoldness, M_1, formed by the curve C and the tangent planes of the developable along the curve, is said to be a characteristic manifoldness of the equation (3) in the case when the equation (3) and

$$dp = rdx + sdy, \quad dq = sdx + tdy \tag{6}$$

reduce to two distinct equations in the three second deriva-

* The following memoirs have been used in the construction of this chapter :

Monge : "Mémoire sur le calcul intégral des équations aux différences partielles," *Histoire de l'Académie des Sciences* (1784).

Ampère : "Mémoire contenant l'application de la théorie exposée dans le xvii^e cahier, etc.", *Journal de l'École Polytechnique*, vol. 18 (1820).

Boole : "Ueber die partielle Differentialgleichung 2. Ordnung, $Rr + Ss + Tt + U(rt - s^2) = V$," *Crelle's Journal*, vol. 61 (1863).

Bour : "Sur l'intégration des équations différentielles partielles du premier et du second ordre," *J. de l'Éc. Poly.*, vol. 22 ; De Morgan : *Cambridge Philosophical Transactions*, vol. 9.

Imschenetsky : "Étude sur les méthodes d'intégration des équations aux dérivées partielles du second ordre d'une fonction de deux variables indépendantes" (traduit du russe par Hoüel).

Graindorge : Mémoires sur l'intégration des équations aux dérivées partielles des deux premiers ordres," *Mémoires de la Société Royale des Sciences de Liège*, 2d series, vol. 5 (1873).

Sophus Lie : "Neue Integrations-methode der Monge-Ampère'schen Gleichung," *Archiv for Mathematik og Naturvidenskab*, vol. 1 (1876) ; "Ueber Complexe, etc.", *Math. Ann.*, vol. 5 (1872).

Darboux : "Mémoire sur les solutions singulières, etc.", *Mém. des sav. étr.*, vol. 27 (1883), pp. 205–238 ; "Théorie des surfaces," vol. 3, p. 263 ff.

tives r, s, t leaving one to be taken arbitrarily. Every equation of the form (3) admits of two systems of characteristic manifoldness, each system depending on an arbitrary function of one variable. The property of characteristics that through every point of an integral surface of (3) there passes one characteristic of each system situated wholly upon the surface is responsible for the capital rôle of these manifoldnesses in the theory of equations (3). It follows that every integral surface of (3) together with its tangent planes forms a two dimensional manifoldness M_2 which is the locus of the characteristic manifoldnesses of (3) and conversely a locus of characteristics is an integral surface. The phraseology of Lie's geometry of surface elements would have been useful in presenting these details. In order to form the equations of the characteristics it is only necessary to express that the system of these equations (3) and (6), linear in r, s, t, reduces to two distinct equations.

Lie's interpretation of the integration problem of a first order equation* leads to a corresponding enlargement of the ordinary definition of an integral of a second order equation ; Goursat follows this lead in order to give greater generality to the theory. If the defining equations of one of the systems of characteristics of (3) be written in a form to include all cases :

$$\left.\begin{aligned} &dz - pdx - qdy = 0, \\ &F(x, y, z, p, q;\ dx, dy, dp, dq) = 0, \\ &F_1(x, y, z, p, q;\ dx, dy, dp, dq) = 0, \end{aligned}\right\} \tag{7}$$

F and F_1 being linear and homogeneous functions of dx, dy, dp, dq, then by an integral of (3) is meant an element manifoldness, M_2, such that through every element of this manifoldness there passes one characteristic manifoldness M_1, satisfying the equations (7) and all of whose elements belong to M_2. With this new definition of an integral the second order equation itself falls in the background while the linear equations in dx, dy, dp, dq which define the characteristics assume the title rôle. Observing that equations

* According to Lie to integrate the first order equation

$$F(x, y, z, p, q) = 0 \tag{a}$$

is to find all families of ∞^2 surface elements, the coördinates (x, y, z, p, q) of whose elements satisfy (a) and the equation

$$dz - pdx - qdy = 0.$$

A surface element is the ensemble of a point (x, y, z) and a plane (of angular coefficients p, q) passing through the point.

(7) are changed into a similar system by a contact transformation we have at once the fundamental property of the equations of Monge and Ampère—when a contact transformation is applied to one of these equations there results an equation of the same form, and the characteristics are changed into the characteristics of the new equation. The above notions are illustrated in the text by two examples relating to surfaces of a complex and the so-called surfaces of translations.

In presenting the method of integration of Monge Goursat remarks that the method of Monge and Ampère consists essentially in a search for integrable combinations of the equations which define one of the systems of characteristics of (3). When the equations of the characteristics of one of the systems admit of two integrable combinations, the method of Monge refers the solution of Cauchy's problem to the integration of a system of ordinary equations. The method of Ampère is more elastic and has the advantage of applicability to equations which the method of Monge fails to integrate. Both methods lead to the same calculations for the integrations of those equations to which both methods are applicable. In the third and fourth part of his memoir Ampère shows that, when the equations of one of the systems of characteristics or of the two systems simultaneously present a single integrable combination, the given equation can be referred to another in which there appear but one or two derivatives of the second order.

The question as to whether the equations of one of the systems of characteristics of (3) admit of integrable combinations identifies itself with that of finding intermediary integrals of the first order which contain an arbitrary constant. The determination of intermediary integrals reduces itself to the problem of finding the integrals common to two linear equations of the first order; the author assumes that the reader is familiar with the latter theory.

The general search for intermediary integrals and the examinations of the different cases in which intermediary integrals exist are greatly aided by the theorem that two intermediary integrals belonging to two different systems of characteristics are always in involution; when the two systems of characteristics are identical any two intermediary integrals are in involution.

If one of the systems of characteristics admits of three integrable combinations* the two systems of characteristics

* The equations of the second order for which the differential equations of the characteristics admit of three integrable combinations were first determined by Lie: *Archiv for Math. og Naturv.*, vol. 1.

are identical. For this case we have the following practical rule : If the three distinct integrable combinations are $du = 0$, $dv = 0$, $dw = 0$ we eliminate p and q from three equations $u = a$, $v = b$, $w = c$ and thus obtain a family of surfaces depending on three parameters a, b, c. The general integral of the given equation is found by establishing in these three parameters two relations of arbitrary form and then taking the envelope of the surfaces thus obtained. When the two systems of characteristics are identical they cannot admit of two integrable combinations. The one remaining case of one integrable combination will not yield to the method of Monge. Ampère succeeded by a very complicated calculation in showing that in this last case the given equation can be referred to one whose only second derivative is r. By Lie's theory of contact transformations this theorem drops out at a stroke. Goursat illustrates these different cases by examples. Among other points in this connection he solves Cauchy's problem for the equation $s = 0$ and demonstrates Lie's theorem already alluded to that every equation of the form (3) can be reduced to a canonical form, say $rt - s^2 = 0$, or $r = 0$, by contact transformations.

When the two systems of characteristics of the equation (3) are distinct, each system admits of two integrable combinations at most. In case this maximum number is attained the given equation admits of two intermediary integrals each containing an arbitrary function. The solution of Cauchy's problem in this instance is referred to quadratures wherein the arbitrary functions need not occur under the sign of integration ; in proof of the latter theorem Goursat employs a lemma from the theory of contact transformations, in fact the direct discussion leads to Lie's theorem:—* All equations of the second order which admit of two intermediary integrals of the first order each containing an arbitrary function and belonging to different characteristics can be reduced by contact transformations to the form $s = 0$. When one system admits of two integrable combinations and the other of but one, the problem of Cauchy is referred to the integration of an ordinary differential equation of the first order ; in general it is impossible to obtain formulæ for the general integral in which the arbitrary functions do not occur under the sign of integration. If one of the systems admits of two integrable combinations and the other of none, Cauchy's problem is referred to the integration of a system of ordinary differen-

* This theorem has also been obtained by Darboux ; see memoir cited at the beginning of this chapter.

tial equations. If neither system admits of two integrable combinations the method of Monge will not give the general integral. If each system admits of but one integrable combination the given equation is reducible to the simple form

$$s - \lambda(x, y, z, p, q) = 0$$

by contact transformation. If only one of the systems possesses an integrable combination a contact transformation reduces the equation to the simple form

$$Hr + 2Ks + M = 0.$$

When the differential equations of the characteristics do not admit of integrable combinations the methods in hand fail to reduce the given equation to a simpler form. Imschenetsky has remarked however that, when we know an integral containing three arbitrary constants, we can derive from the given equation by a contact transformation an equation in which the term $rt - s^2$ does not appear. Goursat gives an *a priori* demonstration of this theorem.

The chapter is concluded by Ampère's integration of the equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

of minimal surfaces, together with a generalization of this method for the equations

$$Hr + 2Ks + Lt = 0,$$

and the solution of Cauchy's problem for surfaces of translation drawn from the researches of Lie.*

3. The third chapter presents a number of examples, taken largely from the theory of surfaces. Goursat states that it is not his object to present results in their simplest form but solely to show how the general processes of integration are applied. The first of these examples finds the surfaces of Joachimsthal, namely those surfaces one of whose systems of lines of curvature are plane curves whose planes pass through a fixed point; the second determines the surfaces of Monge, *i. e.*, the surfaces one of whose systems of lines of curvature is situated upon concentric spheres; the third seeks those surfaces whose lines of curvature are plane curves in both systems. The last example introduces, by a theorem of Bonnet, the problem of the spherical representation of Gauss. The problem of finding those surfaces which

* Sophus Lie: "Untersuchungen über Translationsflächen," *Berichte der Königl. Sächs. Gesellschaft der Wissenschaften*, 1892.

admit of a given spherical representation depends on the integration of an equation of the form

$$r = \lambda^2(x, y)t\,;$$

as a fourth example Goursat proposes to find what conditions the function $\lambda(x, y)$ must satisfy in order that the preceding equation can be integrated by the method of Monge. The fifth example integrates the equation

$$X(x)pt + rt - s^2 = 0\,;*$$

the sixth attacks the form

$$qr + (zq - p)s - pzt = 0.\dagger$$

These specified examples are followed by a series which have to do with the derivation and integration of those equations whose characteristics are either asymptotic lines, lines of curvature, or conjugate lines. The applications are concluded by the integration of the equation

$$(r - pt)^2 = q^2rt.$$

This equation possesses historical interest since it is the first one treated by Ampère.‡ Goursat gives the integration not only by the method of Ampère but also by the method of Monge.

The constant use made of contact transformations in the solution of the above problems amply illustrates the importance of Lie's theory in the integration of partial differential equations.

The author gives an extensive list of exercises at the end of this chapter, many of which, he states, are taken from the treatise of Forsyth.

4. The last chapter § of the first volume has to do with the general theory of characteristics and extends the notion

* See Goursat : "Sur une problème relatif à la déformation des surfaces," *American Journal of Mathematics*, vol. 14.

† See Goursat : "Sur une classe d'équations analogues à l'équation de Clairaut," *Bulletin de la Société Mathématique*, 1895.

‡ *Journal de l'Ecole Polytechnique*, vol. 18, pp. 46 et seq.

§ Ampère : "Considérations générales sur les intégrales des équations aux différentielles partielles," *Journ. de l'Éc. Polytech.*, vol. 17.

Bäcklund : "Ueber partielle Differentialgleichungen höherer Ordnung, die intermediäre erste Integrale besitzen," *Mathematische Annalen*, vols. 11 and 13; "Zur Theorie der Charakteristiken der partiellen Differentialgleichungen zweiter Ordnung," *Ibid.*, vol. 13.

Goursat : "Sur une classe d'équations aux dérivées partielles du second ordre et sur la théorie des intégrales intermédiares," *Acta Mathematica*, vol. 19.

to equations of any form whatever. Let

$$F(x, y, z, p, q, r, s, t) = 0 \qquad (8)$$

be an equation of the second order of arbitrary form.

Every system of values of the variables (x, y, z, p, q, r, s, t) is called an element of the second order. By a characteristic manifoldness, or more simply characteristic, of the equation (8) is meant a simply infinite system of elements of the second order satisfying the relations

$$F(x, y, z, p, q, r, s, t) = 0,$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0, \qquad (9)$$

$$dz = pdx + qdy, \quad dp = rdx + sdy, \quad dq = sdx + tdy, \qquad (10)$$

$$\left.\begin{aligned} \left(\frac{dF}{dx}\right) + R\frac{dr}{dx} + T\frac{ds}{dy} = 0, \\ \left(\frac{dF}{dy}\right) + R\frac{ds}{dx} + T\frac{dt}{dy} = 0; \end{aligned}\right\} \qquad (11)$$

where

$$R = \frac{\partial F}{\partial r}, \quad S = \frac{\partial F}{\partial s}, \quad T = \frac{\partial F}{\partial t};$$

$$\left(\frac{dF}{dx}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}r + \frac{\partial F}{\partial q}s,$$

$$\left(\frac{dF}{dy}\right) = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}q + \frac{\partial F}{\partial p}s + \frac{\partial F}{\partial q}t.$$

Since the equation (9) is a quadratic for dy/dx every equation of the form (8) possesses two distinct systems of characteristics; these systems are identical when $S^2 - 4RT = 0$.

Goursat applies these generalities to the Monge-Ampère equation (3). A characteristic of the first order of the Monge-Ampère equation is defined to be a simply infinite system of elements of the first order (x, y, z, p, q) satisfying the three equations

$$Hdpdy + Ldqdx + Mdxdy + Ndpdq = 0,$$

$$Hdy^2 - 2Kdxdy + Ldx^2 + N(dpdx + dqdy) = 0,$$

$$dz - pdx - qdy = 0.$$

An easy reckoning shows that the characteristics of the first order are identical with the characteristic manifoldnesses studied in the second chapter. Every characteristic of the first order belongs, in general, to an infinity of characteristics of the second order depending on an arbitrary constant. If two integral surfaces admit of all the elements of a characteristic of the first order and if they have contact of the second order at one point of this characteristic they have contact of the second order all along the characteristic.

Two simple examples are introduced to show that those equations whose two systems of characteristics are identical must be studied separately. The remark is also thrown out that there are equations besides the Monge-Ampère equations which admit of characteristics of the first order.

The notions element and characteristic can be further generalized by considering derivatives of any order and the series of values which they take along a characteristic. Putting

$$p_{ik} = \frac{\partial^{i+k} z}{\partial x^i \partial y^k},$$

a system of values of the variables

$$(x, y, z, p, q, r, s, t, p_{30}, \cdots, p_{no}, p_{n-1,1}, \cdots, p_{on})$$

is called an element of the nth order ; when it is displaced along a characteristic, the simply infinite series of elements of the nth order is a characteristic of the nth order. The differential equations of an nth order characteristic are established in the same manner as those of a characteristic of the second order. If $S^2 - 4RT$ is not zero then 1° a characteristic of the nth order is contained in an infinity of characteristics of the $(n+1)$th order, depending on an arbitrary constant; 2° an nth order characteristic is contained in an infinity of characteristics of order $n + r$, depending on r arbitrary constants; 3° if two integral surfaces have all the elements of a second order characteristic in common and have contact of the nth order at one point of this characteristic, they have contact of the nth order all along the characteristic. When $S^2 - 4RT$ is equal to zero the results are all different ; for example, a characteristic of the nth order is, in general, contained in but one characteristic of the $(n + 1)$th order.

All the elements of a characteristic of the second order (for which $S^2 - 4RT$ is not zero) belong to an infinity of integral surfaces, depending on an infinity of arbitrary constants. Two characteristics of the second order belonging to two different systems and having an element of the second order in common determine one integral surface and but one.

Goursat applies these theorems to the Monge-Ampère equations having two distinct systems of characteristics with the results that the two just written hold for these equations when "second" is replaced by "first." These Monge-Ampère equations possess the peculiar property that, being given an integral surface and a characteristic on this surface, there exists an infinity of surfaces having contact of the first order with the first all along this curve; *i. e.*, if there exist manifoldnesses M_1 of first order elements which belong to an infinity of integral surfaces, the value of one of the second order derivatives can be chosen arbitrarily at a point of M_1. The question whether there are other equations of the second order possessing this property is answered in the affirmative; the latter equations also admit of first order characteristics.

The study of these two orders of characteristics leads Goursat to the following classification of equations of the second order:

1° The general equations which admit of two different systems of characteristics, both of the second order.

2° The equations which, when x, y, z, p, q are regarded as parameters and r, s, t as current coördinates, represent a ruled surface, not of the second degree, whose generatices are parallel to that of the cone T,* without the tangent plane to this surface being parallel to a tangent plane of T. These admit of two different systems of characteristics, one of the first order, the other of the second.

3° The equations of Monge and Ampère. They have two systems of characteristics, in general distinct, both of the first order.

4° The equations which, with the same conventions, represent a developable surface admitting the cone T as director cone.

It is clear that this classification is invariant under all contact transformations.

After completing the solution of the problem of Cauchy introduced in the first chapter Goursat proceeds to a study of those equations which admit of characteristics of the first order. The theory of characteristics of the first order is intimately associated with that of intermediary integrals. An equation of the second order taken at random does not, of necessity, admit of an intermediary integral. The con-

* The equation of this cone T is $s^2 - rt = 0$.

ditions that an equation admitting of characteristics of the first order admit of an intermediary integral, $V(x, y, z, p, q; a, b)$ depending on two essentially different parameters a and b are found by replacing λ, ρ, μ, ν, respectively, by

$$\frac{\partial V}{\partial x}+p\frac{\partial V}{\partial z},\quad \frac{\partial V}{\partial y}+q\frac{\partial V}{\partial z},\quad \frac{\partial V}{\partial p},\quad \frac{\partial V}{\partial q}$$

in the equations

$$\varphi(x, y, z, p, q;\ \lambda, \mu, \nu, \rho)=0,\quad \psi(x, y, z, p, q;\ \lambda, \mu, \nu, \rho)=0,$$

which equations represent the conditions, homogeneous in λ, μ, ν, ρ, necessary and sufficient that the straight line

$$\lambda+\mu r+\nu s=0,\quad \rho+\mu s+\nu t=0$$

shall be a generatrix of the surface represented by the given equation, when r, s, t are taken as current coördinates and x, y, z, p, q, as parameters.

By seeking all those equations admitting of characteristics of the first order and of such a nature that the two equations to be satisfied by an intermediary integral form a system in involution, Goursat finds a wide class of integrable equations whose systems of characteristics fall together, thus generalizing a property already established for the Monge-Ampère equations. The integration of this class is referred to the integration of systems in involution. The author shows how we may obtain formulæ for the general integral analogous to those obtained for the equations of Monge and Ampère. For the solution of the problem of finding all equations which can be integrated in this manner he refers the reader to his memoir * already cited.

Goursat points out that the fundamental distinction between the two species of characteristics is contained implicitly in the memoir of Ampère referred to at the beginning of the chapter. He concludes the chapter by a rough sketch of those details of Ampère's memoir which bear directly on this statement. At the end of the chapter there appears a valuable list of exercises.

5. The fifth chapter† of the work begins the second volume and takes up the method of Laplace. Consider the linear equation

* *Acta Mathematica*, vol. 19, pp. 285–340.

† Euler: "Institutiones Calculi Integralis," vol. 3.

Laplace: "Recherches sur le calcul intégral aux différences partielles," *Mémoires de l' Académie*, 1773.

Darboux: "Leçons sur la théorie générale des surfaces," vol. 2, chapters 2 ff.

$$s + ap + bq + cz = M, \tag{12}$$

where a, b, c, M are given functions of the independent variables x and y. In order that this equation admit of an intermediary integral depending on an arbitrary function the functions a, b, c ought to satisfy at least one of the equations

$$h = a_x + ab - c = 0, \quad k = b_y + ab - c = 0. \tag{13}$$

In either case the general integral of (12) is obtained by quadratures. In case both are satisfied simultaneously the given equation admits of two different intermediary integrals and can be reduced to the form $s = 0$. In case neither is satisfied the method of Monge cannot find the general integral. We owe to Laplace a method of transformation which, in certain cases, refers the integration of the equation (12) to that of another of the same form possessing an intermediary integral. The presentation of Laplace's method adopted by Goursat is substantially that given by Darboux in the second volume of his theory of surfaces. Without loss of generality M may be put equal to zero in the equation (12) and the equation is accordingly taken in the simpler form.

The substitutions

$$z_1 = q + az, \quad z_{-1} = p + bz \tag{14}$$

define the first and second transformations of Laplace respectively. Goursat remarks that these transformations differ essentially from contact transformations. A contact transformation applied to a second order equation always yields an equation of the second order, while the transformations of Laplace applied to any equation of the second order give, for the new unknown, a system of several equations of order higher than the second. Their success when operating on the equation (12) is due to the particular form of that equation. They can be decomposed into simpler transformations, for example, the first is equivalent to the three

$$u = ze^{\int a dy}, \quad v = \frac{\partial u}{\partial y}, \quad z_1 = ve^{-\int a dy},$$

of which the first and third are applicable to every equation of the second order, while the second is efficient only with equations of a particular form.

Darboux calls the left hand members of the equation (13) which are represented by h and k respectively, and in general are not zero, the invariants of the equation (12) deprived of its right hand member, since they are unchanged by the change of variables

$$x = \varphi(x'), \quad y = \psi(y'), \quad z = \lambda(x, y)z'.$$

If we consider two linear equations which are derivable one from the other by the substitution $z = \lambda z'$ as essentially the same, a linear equation is completely determined when its invariants are known.

Let (E) be a linear equation without a second member; the first transformation of Laplace applied gives an equation of the same form (E_1) and the second (E_{-1}). The invariants of these two equations depend only upon the invariants of (E). Repeating the transformations a single series is reached

$$\cdots, (E_{-2}), (E_{-1}), (E), (E_1), (E_2), \cdots \qquad (15)$$

in which each equation (E_i) is deduced from the equation (E_{i-1}) by the first substitution and from (E_{i+1}) by the second. Two equations of this series are integrated simultaneously, hence if one of them can be integrated, the whole sequence can be. If one of the equation (E_i) has a zero invariant the series terminates with this equation and every member of the series is integrable by quadratures. The general integral of the original equation (E) in this case consists of an expression containing explicitly an arbitrary function and a finite number of its derivatives while a second arbitrary function enters under the integral signs. This second function Y can not in general be made to disappear; if it be zero the general integral has the form

$$z = AX + A_1X' + \cdots + A_iX^{(i)}, \qquad (16)$$

the A's being determinate functions, while X is an arbitrary function, of x, y. Conversely, if (E) admits of a solution of this form the series of Laplace terminates on one side after i operations at most. The expression (16) is said to be of the $(i+1)$th range with regard to x. Similarly, changing the variables x and y, if (E_{-j}) has a zero invariant, the series of Laplace terminates on the other side and the given equation has a particular integral of the $(j+1)$th range in y. If the series terminates on both sides the general integral is of the form

$$z = AX + A_1X' + \cdots + A_iX^{(i)} + BY + B_1Y' + \cdots + B_jY^{(j)}$$

of range $i + 1$ with regard to x and $j + 1$ with regard to y.

To find all linear equations whose general integral is given by the method of Laplace it is necessary and sufficient to find all those sequences of linear equations in which the equations of the sequence are derivable one from another by the method of Laplace, and which terminate on one or both sides. To find those sequences which terminate on one side is easy, but to those terminating on both sides is more difficult. In the solution of the latter problem Goursat applies the method of attack devised by Darboux.

In illustration of the general method of Laplace Goursat takes up again the equation of minimal surfaces. He then establishes the following theorem,* which in a great number of problems in the theory of surfaces enables us to recognize in advance whether Laplace's method is applicable to certain equations :—If in the $n + 1$ linearly distinct integrals of the equation

$$s + ap + bq + cz = 0, \tag{17}$$

there exists a linear and homogeneous relation whose coefficients are functions of but one of the variables x, y, the series of Laplace relative to this equation terminates on one side after $n - 1$ transformations at most. The theorem's utility is shown by applying it to the problem of finding those surfaces one of whose systems of lines of curvature consists of plane curves.

Every linear equation of the form

$$Ar + 2Bs + Ct + Dp + Eq + Fz + G = 0, \tag{18}$$

where $A, B, \cdots, G$ are any functions of x and y can be reduced to the form (17) provided that $B^2 - AC$ is not zero. To make the reduction it is sufficient to take as new variables u and v satisfying the relations

$$u_x + \lambda_1 u_y = 0, \quad v_x + \lambda_2 v_y = 0,$$

in which λ_1, λ_2 are the roots of $A\lambda^2 - 2B\lambda + C = 0$. However, Legendre † has shown that we can apply directly to the equation (18) a method analogous to the method of Laplace, without making the preceding transformation. Goursat gives Legendre's method in the simple form presented by Imschenetsky.

The reasoning of the preceding sections is valid both in

* Goursat: "Sur les équations linéaires et la méthode de Laplace," *American Journal of Mathematics*, vol. 18.

† *Histoire de l'Académie des Sciences*, 1787—"Mémoire sur l'intégration de quelques équations aux différences partielles," §4, pp. 319-323.

imaginary variables and in real variables. When the coefficients of (18) are real functions of x and y and none but real transformations are employed the equations (18) may be classified into three types:

1° If $B^2 - AC$ is positive, the characteristics of the given equations are real, and the equation is said to appertain to the hyperbolic type. A real substitution refers the given equation to the form

$$s + ap + bq + cz + M = 0;$$

by substituting $\lambda z + \mu$ for z, where λ and μ are functions of x and y suitably chosen, M and one of the coefficients may be made to disappear and the equation to take one of the three forms

$$s + ap + bq = 0, \quad s + ap + cz = 0, \quad s + bq + cz = 0.$$

2°. If $B^2 - AC$ is negative, the characteristics are imaginary and the equation is of the elliptic type; it cannot be referred to the preceding canonical form by a real substitution. The canonical form for this type is

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + a\frac{\partial z}{\partial u} + b\frac{\partial z}{\partial v} + cz + M = 0,$$

where u, v, α, β satisfy

$$\frac{\partial u}{\partial x} = \alpha\frac{\partial u}{\partial y} - \beta\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \alpha\frac{\partial v}{\partial y} + \beta\frac{\partial u}{\partial y},$$

$$A\alpha + B = 0, \; A^2\beta^2 = AC - B^2.$$

Again, in this case, the substitution $z = \lambda z' + \mu$ removes M and one of the a, b, c.

3°. When $B^2 - AC$ is zero, the equation has but one system of characteristics, which are necessarily real, and belongs to the parabolic type. If a new system of variables u and v be taken such that $v =$ *constant* represents the characteristics, the equation assumes the form

$$\frac{\partial^2 z}{\partial u^2} + a\frac{\partial z}{\partial u} + b\frac{\partial z}{\partial v} + cz + M = 0;$$

as in the preceding cases the substitution $z = \lambda z' + \mu$ removes M and c; if for the variable u an integral u_1 of the equation be taken, we see that

$$r + bq = 0$$

may be taken as the canonical form of the equation of the parabolic type.

6. * A system of equations of partial derivatives in m functions $z_1, \cdots, z_m$ and n independent variables $x_1, \cdots, x_n$ is said to be integrable if there exist at least one system of functions

$$z_1 = \varphi_1(x_1, \cdots, x_n), \quad \cdots, \quad z_m = \varphi_m(x, \cdots, x_n)$$

such that on the substitution of the φ's for the z's in the given equations the latter become identically zero. If p is the order of the lowest derivative entering into the system and if every equation of order equal to or inferior to p which admits of all the integrals of the system is an algebraic consequence of these equations, the system is said to be completely integrable. † It can happen that a system of m equations containing fewer than m unknowns admits of integrals depending on an infinity of arbitrary constants. Lie calls such systems Darboux systems. The most important class are formed by the Darboux systems whose general integral possesses the highest degree of generality possible, or systems in involution. Goursat gives a more precise definition of systems in involution in a later paragraph.

Among the systems of particular interest are those in which all the derivatives of a certain order of the unknown functions can be expressed by means of the variables, the unknown functions, and derivatives of a lower order. It is always possible ‡ to recognize, by differentiations and algebraic operations, when such a system is compatible, or to find the finite number of constants on which the general integral of such a system depends. These systems have been extensively studied by Lie. §

Before proceeding to the consideration of certain particular types of systems Goursat makes use of some notions introduced by Lie. The idea of an element of the nth order

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* This chapter, devoted to systems in involution, has been constructed to a large extent from the memoirs of König: *Mathematische Annalen*, vol. 22; Hamburger: *Crelle*, vols. 81, 93, 110; Lie: *Berichte der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig*, 1895; Méray and Riquier: *Annales de l'École Normale*, 1980; Riquier: *Annales de l'Ecole Normale*, 1893; *Mémoires des Savants étrangers*, vol. 33; Bourlet: Thèse de Doctorat, Paris, 1891; Beudon: Thèse de Doctorat, Paris, 1896; von Weber: *Sitzungsberichte der Akad. der Wissenschaften*, München, 1895; Tresse: Thèse de Doctorat, Paris, 1893; Delassus: *Annales de l'École Normale*, 1896, 1897.

† Complétement intégrable, unbeschränkt integrabel.

‡ See Bourlet: Thèse de Doctorat, 1[re] partie, Paris, 1891.

§ Lie: "Theorie der Transformationsgruppen," vol. 1, chapter 10.

was presented in the fourth chapter of the first volume. Two nth order elements infinitely near to each other

$$(x, y, z, \cdots, p_{ik}, \cdots), \quad (x + dx, y + dy, z + dz, \cdots, p_{ik} + dp_{ik}, \cdots)$$

are said to be associated when we have

$$dz = p_{10}dx + p_{01}dy, \quad dp_{i,k} = p_{i+1,k}dx + p_{i,k+1}dy, \quad i + k \leqq n - 1.$$

It is clear that a surface determines a doubly infinite number of associated elements of the nth order; a curve on this surface determines a simply infinite number of these associated elements; such a manifoldness is called an orientation of nth order elements and the curve considered is called the support of the orientation. Lie's notion of an integral for the case of the system

$$F_1 = 0, \quad F_2 = 0, \cdots, F_\mu = 0,$$

where the highest derivative is of the nth order, becomes: every doubly infinite system of associated nth order elements verifying those equations is an integral of the system.

After seeking the integrals common to two equations, one of the first and the other of the second order, Goursat proposes to study the systems of the form

$$r + f(x, y, z, p, q, s) = 0, \quad t + \varphi(x, y, z, p, q, s) = 0.$$

Lie calls such a system a system in involution when the conditions, using the notation of the text,

$$1 - \frac{\partial f}{\partial s}\frac{\partial \varphi}{\partial s} = 0, \quad \left(\frac{df}{dy}\right) - \frac{\partial f}{\partial s}\left(\frac{d\varphi}{dx}\right) = 0, *$$

are satisfied.

A system of this species admits of an infinite number of integrals depending on an infinite number of arbitrary constants.

The theory of systems in involution presents numerous analogies with the theory of partial differential equations of the first order. Every equation of the first order has a family of characteristic manifoldnesses of the first order depending on three parameters; similarly every system in involution

* The operators here are thus defined:

$$\left(\frac{d}{dx}\right) \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial z}p - \frac{\partial}{\partial p}f + \frac{\partial}{\partial q}s,$$

$$\left(\frac{d}{dy}\right) \equiv \frac{\partial}{\partial y} + \frac{\partial}{\partial z}q + \frac{\partial}{\partial p}s - \frac{\partial}{\partial q}\phi.$$

possesses a family of characteristics of the second order depending on five parameters. Each characteristic of the second order contains a characteristic curve which serves as its support, and in general, these characteristic curves themselves depend on five parameters. The integration of a system in involution is referred to that of a system of ordinary differential equations by a method which is a generalization of the method of Cauchy for the integration of a partial differential equation of the first order. The notion of a complete integral of a first order equation finds also a parallel in that of a complete integral of a system in involution, but with this difference, as Goursat points out by means of an example happily chosen—every family of surfaces depending on two parameters is a complete integral of a differential equation of the first order, but while a family of surfaces depending on four parameters gives a complete integral* of a system of two equations of the second order, the latter are not, in general, in involution. That is, a system of two equations of the second order can admit of an infinity of integrals, depending on an arbitrary function, without being in involution. In order that the system be in involution it is necessary that through a curve taken arbitrarily there pass an infinity of integrals depending on an arbitrary constant. After a short study of linear systems in involution and a number of useful examples Goursat concludes this part of the study with the observation that a system of two equations of the second order can admit of an integral depending on an infinity of arbitrary constants in two cases: (1) when the two equations are in involution; (2) when they have a common intermediary integral of the first order.

Two second order equations of any form whatever are said to be in involution when the four equations formed by differentiating the two with regard to x and y reduce to three distinct equations. If the system be formed of one equation of the second order and one, $\varphi = 0$, of the nth order, in order that the two be in involution it is necessary that the latter satisfy one of two sets of conditions.

The determination of the characteristics of a system in involution is the essential part of the problem. Just as the method of Monge and Ampère was chiefly concerned in finding integrable combinations of the differential equations of the characteristics of the first order, we find here a generalization of this method which seeks integrable combinations

* The theory of the complete integral can be extended to systems in involution composed of an equation of the second order and one of the nth order, as Goursat observes in a later remark, p. 94, vol. 2.

of the differential equations of the nth order characteristics. Without loss of generality the given second order equation may be taken in the form $r + f = 0$; * in order to obtain the equations of the characteristics of the system in involution, $r + f = 0$, $\varphi = C$, it is only necessary to adjoin the relation, $\varphi = C$, to the equations of the characteristics of $r + f = 0$. A fundamental property of systems in involution is that every orientation of nth order elements belonging to two equations, one of the second and the other of the nth order in involution, determines a common integral of the system. Of the two equations above, whatever be the constant, C, the function is an invariant † of one of the systems of characteristics of the equation $r + f = 0$.

After considering a system formed of $r + f = 0$ and two equations of any order whatever, and extending various results of the preceding investigation, Goursat presents the ingenious method of Lie ‡ for demonstrating the capital principle which dominates the theory of systems in involution, namely, that every integral is a locus of element manifoldnesses which depend on a finite number of constants, and which, consequently, can be obtained by the integration of a system of ordinary differential equations.

After presenting a remark of Tannenberg's, § connecting systems in involution with Lie's transformation groups and certain systems of total differential equations studied by Hamburger, || Goursat concludes the chapter with some theorems of König ¶ on systems completely integrable. A semi-linear equation, $r + f = 0$, and an equation of the second order, $u = a$, form a system completely integrable if u satisfies a certain equation of the second order. This theorem is generalized for the system formed of a semi-linear equation and one of any order whatever, resulting again in a single equation of condition.

7. The seventh chapter deals with the method of Darboux which appeared in 1870 together with the extensions and applications of this method which have been developed in

* Lie has proposed that equations of this form be called *semi-linear*, a designation which Goursat gives later in the text.

† By an invariant of the nth order of a system of characteristics is meant a function $\phi\,(x, y, z, p_{10}, \ldots, p_{0,n})$, containing at least one of the derivatives $p_{1,n-1}, p_{0,n}$ and no derivative of higher order, which preserves a constant value when it is displaced along a characteristic of the system.

‡ Lie: "Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung," *Berichte der Königl Sächs. Gesell. der Wiss. zu Leipzig*, 1895.

§ *Comptes Rendus*, vol. 120, p. 674.

|| *Crelle*, vol. 110.

¶ *Mathematische Annalen*, vol. 24.

the memoirs* of a number of geometers since that date. Goursat presents a succinct résumé of the theorems given in Darboux's memoir quoting largely from the memoir itself, and calls attention to the fact that a number of the theorems have already been demonstrated in the sixth chapter of the present work. He then proceeds to consider with Darboux the case where each of the two systems of characteristics admits of two integrable combinations, so that the second order equation, $F=0$, admits of two distinct intermediary integrals belonging to the two different systems of characteristic and not necessarily of the same order,

$$u-\varphi(v)=0, \quad u_1-\psi(v_1)=0.$$

Every integral of the given equation satisfies two equations of this form, and conversely, whatever be the functions φ and ψ, the three simultaneous equations

$$F=0, \quad u-\varphi(v)=0, \quad u_1-\psi(v_1)=0,$$

form a system which is completely integrable, whose general integral, which depends on a finite number of constants, can be obtained by the integration of a system of ordinary differential equations.

In order to integrate this system two auxiliary variables α and β are introduced by putting

$$v=\alpha, \quad u=\varphi(\alpha), \quad v_1=\beta, \quad u_1=\psi(\beta);$$

by means of these four equations and the equation, $F=0$, five of the eight variables x, y, z, p, q, r, s, t can be written as functions of the other three and $\alpha, \beta, \varphi(\alpha), \psi(\beta)$. By replacing those five variables by their values in

$$dz=pdx+qdy, \quad dp=rdx+sdy, \quad dq=sdx+tdy,$$

we are led to a system of total differential equations, which is completely integrable, for determining the other three variables in functions of α and β.

* Darboux : *Comptes Rendus*, vol. 70, *Ann. de l'Ec. Norm.*, 1st series, vol. 7, 1870 ; Falk : *Nova Acta Regiae Soc. Ups.*, 3d series, vol. 8, 1872; Picard *Comptes Rendus*, vol. 78 (1874) ; Hamburger : *Crelle*, vol. 81 (1876), vol. 93 (1882) ; Winckler : *Sitzungsb. d. k. Akad. d. Wiss., Wien, Abth.* II., vols. 88, 89 (1883–84) ; 1883–84 ; König : *Math. Ann.* vol. 24, (1884) ; Sersawy : *Denkschriften d. k. Akad. d. Wiss., Wien, Math. Naturw. Classe*, vol. 49 (1885); Boer: *Archives néerlandaises*, vol. 27 ; Speckman : *Archives néerlandaises* vol. 28 ; Lévy : *Comptes Rendus*, vol. 25 (1872) Lie : *Archiv for Math. og Naturw.*, vol. 5 (1889) ; *Christiania Forhandlinger*, 1880, *Leipz. Berichte*, 1895; Beudon: Thèse de Doctorat, 1896; Weber: *Math. Ann.* vol. 47, *Sitzungsb. d. Math. Phys. Classe d. k. Bayer. Akad. d. Wiss.* vols. 25, 26; Sonine: *Bull. d. Sc. Math.*, 1st series, vol. 10.

When u, v, u_1, v_1 contain only derivatives of the first order, the given equation is necessarily of the form of the equations of Monge and Ampère. The next simplest case is when u, v, u_1, v_1 contain no derivative higher than the second.

Goursat applies the preceding method to several forms by way of illustration, among which are Liouville's equation $s = e^z$, and the defining equations of minimal surfaces and surfaces of translation.

The solution of Cauchy's problem for the semi-linear equation $r + f = 0$, when but one of the systems of characteristics of the nth order possesses two distinct integrable combinations $du = 0$, $dv = 0$ is referred to the integration of two consecutive systems of ordinary differential equations. The first of these systems which gives u and v does not depend on the initial conditions, but the second one is dependent on the initial conditions.

The method of Darboux is again applicable to the third case, namely, when the two systems of characteristics coincide. After comparing the method for this case with that of Ampère, Goursat develops a number of theorems on invariants for use in subsequent sections. If $f = 0$ admits of an nth order intermediary integral $u - \varphi(v) = 0$, then u and v are invariants of one of the systems of characteristics of $f = 0$. If u and v are two distinct invariants of the same system of characteristics, then du/dv is a new invariant. When the two families of invariants are distinct, there exists at most in each family one distinct invariant of the nth order for every value of n greater than 2 ; each of the systems admits of at most two invariants of the first order ; if this maximum number is attained the equation must belong to those of Monge and Ampère ; each distinct system of characteristics has at most three invariants of the first or second order ; if one of the distinct systems admits of more than one invariant, it admits of an infinite number among which there can always be found two, u and v, such that all are expressed in the following sequence

$$u,\ v,\ v_1 = \frac{dv}{du},\ v_2 = \frac{dv_1}{du},\ \cdots,\ v_n = \frac{dv_{n-1}}{du},\ \cdots.$$

Applying these theorems to the solution of the problem of finding all equations integrable by the method of Darboux and possessing two coincident systems of characteristics, Goursat finds that the only equations of this category are those* already integrated in chapter four of the first vol-

Acta Mathematica, vol. 19.

ume of this treatise, and which include as a particular case those equations of Monge and Ampère in which the differential equations of the first order characteristics admit of three distinct integrable combinations.

For the case of linear equations the method of Darboux and the method of Laplace succeed simultaneously; for the most general linear form Darboux's method leads to the same results as Legendre's generalization of the method of Laplace.

Among the examples used in this chapter is Lie's* elegant discussion of the equation $s = f(z)$. This equation is integrable by the method of Darboux only in the case when

$$f'^2(z) - f(z)f''(z)$$

is different from zero.

An equation of the second order which is integrable by the method of Darboux possesses the following property: every integral of the equation belongs to another partial differential equation which has an infinity of integrals depending on an arbitrary function in common with the given equation, without admitting of all of them. This property makes possible the application of Lie's theory of infinite transformation groups to the problem of constructing equations integrable by the method of Darboux. This application Lie has made in a masterly manner in a recent memoir† already cited. Goursat presents an exposition of the method confining himself to point transformations.

The formulæ

$$x_1 = P(x, y, z), \quad y_1 = Q(x, y, z), \quad z_1 = R(x, y, z), \quad (19)$$

which make the point (x_1, y_1, z_1) correspond to the point (x, y, z) define a point transformation of ordinary space. If the functions P, Q, R depend upon a finite number of arbitrary parameters, or on one or more arbitrary functions, the preceding equations define an ensemble or family of point transformations. Such an ensemble forms a group when the sequence of any two transformations of the ensemble gives a transformation belonging to the same ensemble. When the functions P, Q, R depend on one or several arbitrary functions the group is said to be an *infinite* group or of the infinite order.

* Lie: "Discussion der Differentialgleichung $s = f(z)$," *Archiv for Math. og Natur.*, vol. 6, Christiania (1880).

† Lie: "Zur allgemeinen Theorie * * *," *Leipz. Berichte*, 1895.

Thus for example the formulæ

$$x_1 = x, \quad y_1 = y, \quad z_1 = Z(z), \tag{20}$$

define an infinite group depending on the arbitrary function $Z(z)$.

Designating by $p_1, q_1, r_1, s_1, t_1, \cdots$ the partial derivatives of z_1 with regard to x_1 and y_1, the equations (19) give the following for $p_1, q_1, \cdots$, as functions of $x, y, z, p, q, \cdots$,

$$p_1 = f_1(x, y, z, p, q), \quad q_1 = f_2(x, y, z, p, q), \tag{21}$$
$$r_1 = \varphi_1(x, y, z, p, q, r, s, t), \quad s_1 = \varphi_2(x, y, z, p, q, r, s, t),$$
$$t_1 = \varphi_3(x, y, z, p, q, r, s, t).$$

When the equations (19) define a group, the equations (19) and (21) also define a group which is called the extended* group of the original group (19).

An equation of the second order $F = 0$ is said to admit of the group of point transformations (19) when the equation is invariant by all the transformations of the extended group.

For example, the second extension of the group (20) is given by the equations (21) and the following:

$$p_1 = Z'p, \quad q_1 = Z'q, \quad r_1 = Z'r + Z''p^2, \quad s_1 = Z's + Z''pq,$$
$$t_1 = Z't + Z''q^2;$$

the elimination of Z' and Z'' yields

$$\frac{q_1}{p_1} = \frac{q_1}{p_1}, \quad \frac{q_1r_1 - p_1s_1}{q_1^2} = \frac{qr - ps}{q^2}, \quad \frac{q_1s_1 - p_1t_1}{q_1^2} = \frac{qs - pt}{q^2},$$

which show that every equation of the form

$$F\left(x, y, \frac{p}{q}, \frac{qr - ps}{q^2}, \frac{qs - pt}{q^2}\right) = 0 \tag{22}$$

admits of the group (20).

Consider then the general case of a second order equation admitting of an infinite group depending on a single arbitrary function. Let us suppose that the equations (19) of this group contain explicitly an arbitrary function and its derivatives up to a determinate order. According to Lie's general theories, this group possesses an infinity of differential invariants, that is, functions $u(x, y, z, p, q, \cdots)$ con-

* Erweiterte Gruppe, groupe prolongé.

taining derivatives of z up to as high an order as we wish, which are reproduced by all the transformations of the group. Let u, v, w be three distinct differential invariants such that the equation $F=0$ cannot be taken in the form $\Phi(u, v, w)=0$. Let (S) be any integral of $F=0$; choose $\Phi(u, v, w)$ in such a manner that this integral (S) satisfies the relation $\Phi=0$. The two equations

$$F=0, \quad \Phi(u, v, w)=0,$$

admit of all the transformations of the group and having already an integral (S) in common, they have an infinity of integrals in common depending on an arbitrary function, namely, those deduced from S by all the transformations of the group. Every integral of the given equation of the second order belongs then to another equation which has an infinity of integrals in common with $F=0$, depending on an arbitrary function, without admitting of all of them. The equation $F=0$ is then integrable by the method of Darboux.

Take, for example, the equation (22) which admits of the group (20); x, y, $\frac{p}{q}$ are three differential invariants of this group. Every integral of the equation (22) then verifies an equation of the first order

$$\Phi\left(x, y, \frac{p}{q}\right)=0$$

which admits of infinity of integrals of $F=0$, depending on an arbitrary function. It is necessary for this that the equation of the first order be an intermediary integral of the equation of the second order, and that the latter admit of an intermediary integral of the first order depending on an arbitrary function. It is easy to verify this for the equation (22), since putting $u=\frac{p}{q}$, that equation becomes

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0.$$

In the same way infinite groups of contact transformations* may be used to design differential equations integrable by the method of Darboux. Goursat gives an ex-

* A contact transformation of ordinary space is a transformation of the variables (x, y, z, p, q) into the variables $(x_1, y_1, z_1, p_1, q_1)$ which leaves invariant the Pfaffian equation $dz-pdx-qdy=0$.

ample * to show that these two great categories of infinite groups do not determine all equations integrable by Darboux's method.

By extending to an equation of the second order the method of Lagrange and Charpit for the equation of the first order, König has developed a method of integration which does not differ essentially from Darboux's method. Goursat concludes the chapter with an exposition of the processes employed by König.†

8. Ampère ‡ gave as a necessary condition that an integral of the equation $r + f = 0$ be a general integral the impossibility of deducing, from the definition of this integral, any relation of equality in the variables $(x, y, z, p_{1,0}, p_{1,1}, \cdots, p_{1,k-1}; p_{0,1}, \cdots, p_{0,k})$ however great the integer k be taken. Goursat points out that Ampère did not explicitly state that this criterion is sufficient but that in its subsequent use Ampère implied his belief in its sufficiency. On the other hand, an integral is general in Cauchy's sense of the word if the two series

$$\Sigma \frac{(p_{0,k})_0}{k!}(y - y_0)^k, \quad \Sigma \frac{(p_{1,k-1})_0}{(k-1)!}(y - y_0)^{k-1},$$

are convergent for values of $y - y_0$ whose moduli do not exceed a certain limit, where (x_0, y_0) can vary arbitrarily within a limited domain of holomorphism.

An integral general in the sense of Cauchy is necessarily general in the sense of Ampère, but the converse is not true. For this reason, in order to give a solid base to the theory of partial differential equations, Goursat adopted in the first volume the definition of a general integral which Darboux deduced from the works of Cauchy.

In the memoir of Ampère equations admitting of a general integral of the first class are studied. Ampère did not express in clear manner exactly what the phrase "first class" is to signify, but Goursat has gleaned the following precise definition of this class of equations from the memoir: An equation of the second order, $F = 0$, is said to admit of a general integral of the first class, or simply, to be an equation of the first class, if it is possible to obtain as representa-

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* The equation $(x + y)^2 s^2 - 4pq = 0$ is integrable by the method of Darboux and yet it does not admit of an infinite group of transformations.

† *Mathematische Annalen*, vol. 24.

‡ "Considérations générales sur les intégrales des équations aux différentielles partielles," *Journ. de l'Éc. Polytech.*, vol. 17.

tives of the general integral of the equation, equations of the following form:

$$\begin{aligned} x &= V_1[\alpha, \beta, f_1(\alpha), f_1'(\alpha), \cdots, f_2(\alpha), f_2'(\alpha), \\ &\quad \cdots, \varphi_1(\beta), \varphi_1'(\beta), \cdots], \\ y &= V_2[\alpha, \beta, f_1(\alpha), f_1'(\alpha), \cdots, f_2(\alpha), f_2'(\alpha), \\ &\quad \cdots, \varphi_1(\beta), \varphi_1'(\beta), \cdots], \\ z &= V_3[\alpha, \beta, f_1(\alpha), f_1'(\alpha), \cdots, f_2(\alpha), f_2'(\alpha), \\ &\quad \cdots, \varphi_1(\beta), \varphi_1'(\beta), \cdots], \end{aligned} \tag{23}$$

where V_1, V_2, V_3 are determinate functions of two auxiliary functions α and β, of p functions of $\alpha, f_1(\alpha), f_2(\alpha), \cdots, f_p(\alpha)$, of q functions of $\beta, \varphi_1(\beta), \varphi_2(\beta), \cdots, \varphi_q(\beta)$, and of a finite number of their derivatives, the p functions of $\alpha, f_1, \cdots, f_p$ being subjected to the condition of verifying $p-1$ differential equations of any order whatever, the q functions of $p, \varphi_1, \cdots \varphi_q$ having to verify $q-1$ differential equations of any order whatever; the formulæ (23) then contain in reality only two arbitrary functions, one arbitrary function of α and one arbitrary function of β.

At the close of his memoir, previously cited, Darboux announces that his method is applicable to all equations of the first class. Goursat proves this theorem by proving that the theorem can be extended to equations possessing an integral more general than that given by the equations (23), namely, to the equation whose general integral is represented by formulæ such as

$$\begin{gathered} x = V_1[\alpha, \beta, f(\alpha), f'(\alpha), \cdots, f^{(m)}(\alpha), \varphi(\beta), \varphi'(\beta), \cdots, \\ \varphi^{(n)}(\beta), F_1, F_2, \cdots, F_k], \ y = V_2, \ z = V_3, \end{gathered}$$

where the F_i are defined by the system of total differential equations

$$\begin{gathered} dF_i = \Phi_i[\alpha, \beta, f(\alpha), \cdots, f^{(m)}(\alpha), \varphi(\beta), \cdots, \varphi^{(n)}(\beta), F_1, \cdots, \\ F_k]d\alpha + \Pi_i[\alpha, \beta, f(\alpha), \cdots, f^{(m)}(\alpha), \varphi(\beta), \cdots, \\ \varphi^{(n)}(\beta), F_1, \cdots, F_k]d\beta, \ (i = 1, \cdots, k) \end{gathered}$$

which are supposed to be completely integrable whatever the arbitrary functions $f(\alpha)$ and $\varphi(\beta)$ are.

The study of the method of Darboux suggests a great number of subjects for investigation. For example, it would be very interesting to know all the equations of the second order to which the method of Darboux is applicable with success. For those equations whose two systems of characteristics coincide, the problem has been completely solved (the solution is given in the preceding chapter); but the solutions of the general case which we now possess are extremely special. Goursat proposes to investigate the particular problem of finding all those equations which have an *explicit* general integral, *i. e.*, such that x, y, and z can be expressed as determinate functions of two auxiliary variables α, β, of an arbitrary function of α, of an arbitrary function of β, and of a finite number of their derivatives. This problem is referred to one proposed by Moutard,* whose solution is the same as that of finding all linear equations for which the series of Laplace terminates in both directions.

Goursat observes that the investigations of the treatise show that the method of Monge, that of Laplace, and those methods more or less particular proposed for special forms, are all included in the general method of Darboux, which appears thus as the most powerful means we possess of integrating a partial differential equation of the second order, or, putting it more precisely, of referring the problem of integration to that of integrating one or more systems of ordinary differential equations. In 1872, Maurice Lévy announced † without demonstration, a necessary and sufficient condition for obtaining the general integral of a partial differential equation of the second order by means of the integration of k successive systems of ordinary differential equations, each containing any number of equations with an equal number of unknowns; this criterion is the same as that deduced by the application of Darboux's method. Lévy's theorem remained undemonstrated until very recently. E. von Weber ‡ has given a demonstration with which, with slight modifications, Goursat concludes this chapter.

9. Lie's theory of contact transformations plays a most important part in the theory of partial differential equations of the first order; to integrate a first order equation having a single unknown function and any number of independent variables, is to find a suitable contact transformation which

* *Comptes Rendus*, vol. 70, p. 834; the theorems given by Moutard have been demonstrated by Cosserat in a note added to the fourth volume of Darboux's "Théorie générale des surfaces."

† *Comptes Rendus*, vol. 75, p. 1094.

‡ E. von Weber: "Ueber partielle Differentialgleichungen II. Ordnung, die sich durch gewöhnliche Differentialgleichungen integrieren lassen," *Sitzungsb. d. math.-phys. Classe d. k. bayer. Akad. d. Wiss.*, vol. 26 (1896), pp. 425–437.

will reduce the equation to an immediately integrable form, say $p_1 = 0$. Contact transformations have not so wide an application to partial differential equations of the second order. In certain cases an equation of the second order can be referred to a simpler form by contact transformation, but it is not possible to transform an equation which is not integrable by Darboux's method into one integrable by that method, by contact transformation. On the other hand, when two equations are derivable, one from the other by contact transformation, if the one is integrable by the method of Darboux, the other is also integrable by this method. There are transformations whose successful applicability is conditioned on the particular form of the equation; such, for example, are the transformation of Laplace and that used for integrating Liouville's equation.

The present chapter proposes a review of those transformations most frequently employed, and an attempt to refer them to certain general types. Goursat begins with those equations of the second order in which one of the derivatives of the unknown function verifies a single equation of the second order. The transformation, formed by substituting this derivative as a new unknown, includes various known transformations, in certain cases transforms linear equation into non-linear and conversely, is sometimes applicable repeatedly to the same equation, and leads to a more general transformation of the same kind by combining it with a contact transformation. The author applies this type of transformation to several examples.

The study of a system of two simultaneous equations of the first order in two unknown functions may lead in certain cases, by the elimination of one of the unknown functions, to equations of the second order which are integrable simultaneously. In this event, several correspondences are possible; 1° the integrals of the two resulting second order equations may correspond one-to-one; 2° an integral of the first, say, may correspond to an integral of the second, while to an integral of the second there corresponds an infinity of integrals of the first, depending on an arbitrary constant; 3° to every integral of one equation there may correspond an infinity of the other depending on an arbitrary constant. The author discusses these various possibilities in detail. One remarkable transformation among those thus derived is the well known one which presents itself in the study of surfaces of constant curvature.* In case the

* Darboux: "Théorie générale des surfaces," vol. 3, pp. 432 ff.

two equations of the above system are linear, the adjoined equation * is a useful auxiliary. The adjoined equation has the same invariants as the original equation, taken in the reverse order. This property leads to the transformation known as the transformation of Moutard.† These transformations are cases of more general transformations called the transformations of Bäcklund.‡ The author defines and discusses these transformations and gives Bäcklund's particular example § of such a transformation and also the more recent example given by Cosserat. || He points out that the transformation $q = u$, already studied, and which transforms the equation

$$r + X_1(x)p + X_2(x)z + F(x, y, q, s, t) = 0$$

into the equation

$$\frac{\partial^2 u}{\partial x^2} + X_1 \frac{\partial u}{\partial x} + X_2 u + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial s}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial F}{\partial t}\frac{\partial^2 u}{\partial y^2} = 0,$$

is not a transformation of Bäcklund nor equivalent to a succession of transformations of Bäcklund.

Lie's theory of contact transformations, the investigation of the simultaneous system of two differential equations of the first order containing two unknown functions and the resulting transformations together with the analogous study for the definition of Bäcklund's transformation prepare Goursat to conclude this difficult part of the subject by formulating the following definition of transformations which, when developed, will probably constitute the most

* The *adjoined equation* of the linear equation

$$Ar + Bs + Ct + Dp + Eq + Fz = 0,$$

for example, expresses the condition that the product

$$v(Ar + Bs + Ct + Dp + Eq + Fz)$$

shall have the form

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

where P and Q are functions of x, y, z, p, q. See Darboux:—"Théorie générale des Surfaces," vol. 2, chap. 4.

† *Comptes Rendus*, vol. 70, p. 834 (1870) ; *Journ. de l'Ec. Poly*, vol. 45 (1878).

‡ *Mathematische Annalen*, vol. 17 (1880), vol. 19 (1882).

§ *Lunds Universitets Arsskrift*, vol. 19 (1883).

|| *Comptes Rendus*, vol. 124, pp. 741–744 (April 1897).

powerful transformation-implement yet known in the theory of partial differential equations:

Consider a system of m equations of the first order

$$F_1 = 0, \quad F_2 = 0, \cdots, \quad F_m = 0, \tag{24}$$

in two independent variables x, y and n unknown functions $z_1, z_2, \cdots, z_n (n \leqq m)$. The elimination of all the unknown functions except one of them leads, in general, to several simultaneous equations for determining the latter. Imagine that all the partial derivatives of $z_2, z_3, \cdots, z_n$, starting at a certain order, can be expressed by means of the equations (24) and those derived from them by differentiation, in terms of partial derivatives of a lower order and of the partial derivatives of z_1. If, by writing the conditions of integrability, we are led to a single equation of the second order in z_1

$$G\left(x, y, z_1, \frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y}, \frac{\partial^2 z_1}{\partial x^2}, \frac{\partial^2 z_1}{\partial x \partial y}, \frac{\partial^2 z_1}{\partial y^2}\right) = 0, \tag{25}$$

the integration of the system (24) is referred to the integration of the equation (25). To every integral of this equation there correspond integrals of the system (24), depending on a finite number of arbitrary constants, which are obtained by the integration of a system of ordinary differential equations. Now, it can happen that this reduction of the system (24) to an equation of the second order can be effected in several different ways essentially distinct. For example, imagine that, after a change of variables if necessary, the elimination of the $n - 1$ new unknowns leads to an equation of the second order for the last unknown

$$G_1\left(x', y', \frac{\partial u_1}{\partial x'}, \frac{\partial u_1}{\partial y'}, \cdots\right) = 0; \tag{26}$$

we have thus established a correspondence between the equations (25) and (26) of such a nature that the integration of one carries with it the integration of the other. To every integral of one of them correspond integrals of the other depending on a finite number of constants and obtained by integrating a system of ordinary differential equations. If one of the two equations is integrable by the method of Darboux the other is integrable by the same method.

10. The notion of characteristics extends itself without difficulty to equations of a higher order or to systems of any number of equations, provided the number of indepen-

dent variables is two. Darboux's method of integration, which rests on the theory of characteristics, is capable of a similar extension without other difficulty than that of the increasing complication in the computations. Goursat proposes in the concluding chapter to examine two particularly important cases, that of a unique equation of the nth order and that of a system of n equations of the first order in n unknowns. Hamburger has devoted two important memoirs* to this study; the author follows a different course from that adopted by Hamburger, which he regards as more natural and fruitful. The recent memoirs† of E. von Weber appeared too late to be available in the preparation of this chapter.

The point of departure, as usual, is the problem of Cauchy. This problem generalized for an equation of the nth order consists in determining an integral of this equation admitting of all the elements of a given orientation of $n - 1$ elements and regular in the neighborhood of one of these elements. A study of the characteristics of the nth and higher orders leads to extensions of the theorems already found for those of the second order. There is a great analogy in both the reasoning and the results between the case of the characteristics of second order equation and that of those of an equation of the nth order. But there is also an essential difference. If two characteristics of the different systems of a second order equation have a second order element in common, they belong to the same integral surface. On the contrary, in the case of an equation of order $n > 2$, two nth order characteristics of different systems, having an nth order element in common, do not appertain, in general, to the same integral surface.

The question as to whether an nth order equation admits of $(n - 1)$th order characteristics leads to the result in case of linear equations that an nth order linear equation admits, in general, of n distinct families of $(n - 1)$th order characteristics, which are defined by linear equations in the differentials; then the method of Monge for equations of the second order can be immediately extended to linear equations of the nth order.‡

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* Zur Theorie der Integration eines Systems von n linearen partiellen Differentialgleichungen erster Ordnung mit zwei unabhängigen und n abhängigen Veränderlichen," *Crelle*, vol. 81; "Zur Theorie der Integration eines Systems von n nicht linearen partiellen Differentialgleichungen erster Ordnung mit zwei unabhängigen und n abhängigen Veränderlichen," *Crelle*, vol. 93.

† *Mathematische Annalen*, vol. 79; *Crelle*, vol. 118.

‡ Natani: "Die höhere Analysis," p. 380.

The method of integration of Darboux is applied without difficulty to equations of any order in two independent variables. The method of exposition which Goursat adopts in extending the method of Darboux includes as a particular case the extension of the method of Monge when its extension is possible.

Taking the second problem proposed in the introduction of the chapter Goursat makes a rapid study of a system of n equations of the first order in n unknowns in which the principal steps are the setting of a problem analogous to Cauchy's problem, the definition and determination of the characteristics of the first order, the extension of Monge's method, the study of singular systems and their reduction to a normal form, the consideration of zero order characteristics, of Jacobi's equations,* of linear systems, and of characteristics of a higher order.

The chapter ends with certain generalities on equations having more than two independent variables. All the methods of the book rest upon the consideration of certain one-dimensional manifoldnesses or characteristic manifoldnesses, which possess particular properties relative to a given equation. In order to extend the methods to equations in more than two independent variables it seems most natural then to inquire, first, how this fruitful notion of characteristics can be extended. This extension can be made in several ways according to what property of characteristics is taken as the most important. For example, we can start with the problem of Cauchy generalized as Beudon† has done. Goursat indicates another extension which leads only to very particular forms in three variables. Natani‡ has pointed out a possible extension of the method of Monge to linear equations of the second order having any number of variables. Probably the most recent memoir on this phase of the subject is that of Vivanti.§

The volume is terminated by two notes, one on Darboux's auxiliary equation and its use in the application of Darboux's method and the other on certain theorems relative to the characteristics of simultaneous systems.

EDGAR ODELL LOVETT.

PRINCETON, NEW JERSEY,
25 *April*, 1898.

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* *Crelle*, vol. 2, p. 231.

† *Bulletin de la Société mathematique*, vol. 25, pp. 108–120.

‡ Loc. cit., p. 388.

§ *Mathematische Annalen*, vol. 48 (1897).