substitutions are commutative. Hence the commutator of two such operators is of order 2.*

6. A Hamilton group of order 2^{α} contains $2^{2\alpha-6}$ quaternion groups as subgroups. All of these have the commutator group of the entire group in common.†

CORNELL UNIVERSITY,
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NOTE ON THE INFINITESIMAL PROJECTIVE TRANSFORMATION.

BY PROFESSOR EDGAR ODELL LOVETT.

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IT is proposed here to find the form of the most general infinitesimal projective transformation‡ of ordinary space directly from its simplest characteristic geometric property. Geometrically, infinitesimal projective transformations of space are those infinitesimal point transformations which transform a plane into a plane, *i e.*, which leave invariant the family of ∞^3 planes of ordinary space. Analytically, then, the most general infinitesimal projective transformation is the point transformation

$$Uf \equiv \xi(x, y, z) \frac{\partial f}{\partial x} + \eta(x, y, z) \frac{\partial f}{\partial y} + \zeta(x, y, z) \frac{\partial f}{\partial z} \quad (1)$$

which leaves invariant the partial differential equations

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* Cf. Dedekind: loc. cit.

† Cf. Miller: *Comptes Rendus*, vol. 126 (1898), pp. 1406–1408.

‡ In a note on the general projective transformation, *Annals of Mathematics*, vol. 10, No. 1, the forms of the finite projective transformations of ordinary space and those of n-dimensional space are found directly from the conditions for the invariance of the equations $y'' = 0$, $z'' = 0$, which expresses the geometric property that straight line is changed into straight line by these transformations. The form of the general infinitesimal projective transformation of ordinary space is deduced from the finite transformation by the method of Lie. In this derivation three steps are made to intervene, two of which are removed and the other replaced by a simpler one by the method of the present note: 1° two intersecting planes producing the straight line and its property of invariance; 2° the ordinary differential equations of the straight line and the conditions for their invariance; 3° the finite forms of the transformation.

$$\frac{\partial^2 z}{\partial x^2}=0, \quad \frac{\partial^2 z}{\partial x \partial y}=0, \quad \frac{\partial^2 z}{\partial y^2}=0, \tag{2}$$

of all planes.

The infinitesimal transformation Uf assigns to x, y, z the respective increments

$$\delta x=\xi(x, y, z)\delta\varepsilon, \quad \delta y=\eta(x, y, z)\delta\varepsilon, \quad \delta z=\zeta(x, y, z)\delta\varepsilon, \tag{3}$$

where $\delta\varepsilon$ is an arbitrary infinitesimal of the first order.

In order to determine how Uf changes the quantities

$$p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial^2 z}{\partial x^2}, \quad s \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad t \equiv \frac{\partial^2 z}{\partial y^2}, \tag{4}$$

i. e., to find the increments δp, δq, δr, δs, δt and thus determine what is known as the second extension of the point transformation Uf, namely

$$\begin{aligned} U''f \equiv \xi \frac{\partial f}{\partial x}+\eta \frac{\partial f}{\partial y}+\zeta \frac{\partial f}{\partial z}+\pi \frac{\partial f}{\partial p}+\varkappa \frac{\partial f}{\partial q} \\ +\rho \frac{\partial f}{\partial r}+\sigma \frac{\partial f}{\partial s}+\tau \frac{\partial f}{\partial t}, \end{aligned} \tag{5}$$

we proceed as follows:*

The variation of the identity

$$dz \equiv p dx + q dy \tag{6}$$

gives

$$d\delta z \equiv \delta p \cdot dx + \delta q \cdot dy + p d\delta x + q d\delta y, \tag{7}$$

by virtue of the fact that d and δ are commutative. δx, δy, δz are given functions of x, y, z from (3). The identity (7) is to exist for all values of dx and dy, hence it breaks up into two equations which determine δp and δq.

Similarly the variation of the identities

$$dp \equiv r dx + s dy, \quad dq \equiv s dx + t dy, \tag{8}$$

yields

$$\left.\begin{aligned} d\delta p \equiv \delta r \cdot dx + \delta s \cdot dy + r d\delta x + s d\delta y, \\ d\delta q \equiv \delta s \cdot dx + \delta t \cdot dy + s d\delta x + t d\delta y. \end{aligned}\right\} \tag{9}$$

These two break up into four equations for δr, δs, δt. The computations of these increments result in

* See Lie: Vorlesungen über continuierliche Gruppen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1893, pp. 709–710.

$$\delta p = \pi \delta \varepsilon, \quad \delta q = \varkappa \delta \varepsilon, \quad \delta r = \rho \delta \varepsilon, \quad \delta s = \sigma \delta \varepsilon, \quad \delta t = \tau \delta \varepsilon, \tag{10}$$

where

$$\pi = \zeta_x + p\zeta_z - p(\xi_x + p\xi_z) - q(\eta_x + p\eta_z),$$
$$\varkappa = \zeta_y + q\zeta_z - p(\xi_y + q\xi_z) - q(\eta_y + q\eta_z),$$
$$\rho = \pi_x + p\pi_z + r\pi_p + s\pi_q - r(\xi_x + p\xi_z) - s(\eta_x + p\eta_z),$$
$$\begin{cases} \sigma = \pi_y + q\pi_z + s\pi_p + t\pi_q - r(\xi_y + q\xi_z) - s(\eta_y + q\eta_z), \\ \sigma = \varkappa_x + p\varkappa_z + r\varkappa_p + s\varkappa_q - s(\xi_x + p\xi_z) - t(\eta_x + p\eta_z), \end{cases}$$
$$\tau = \varkappa_y + q\varkappa_z + s\varkappa_p + t\varkappa_q - s(\xi_y + q\xi_z) - t(\eta_y + q\eta_z). \tag{11}$$

The equations (2) are invariant by the transformation (1) when they are invariant by the twice extended transformation (5) ; they are invariant by the latter if

$$\delta r = 0, \quad \delta s = 0, \quad \delta t = 0 \tag{12}$$

for all values of x, y, z, p, q when $r = 0$, $s = 0$, $t = 0$.

The complete computation of ρ, σ, and τ is very much abbreviated by omitting at once the terms involving r, s, t and reckoning out only

$$\pi_x + p\pi_z, \quad \pi_y + q\pi_z, \quad \varkappa_y + q\varkappa_z. \tag{13}$$

The demand that these variations shall be zero for all values of p and q furnishes the following equations for the functions ξ, η, ζ :

$$\xi_{yy} = 0, \quad \xi_{yz} = 0, \quad \xi_{zz} = 0 ; \tag{14}$$

$$\eta_{xx} = 0, \quad \eta_{zx} = 0, \quad \eta_{zz} = 0 ; \tag{15}$$

$$\zeta_{xx} = 0, \quad \zeta_{xy} = 0, \quad \zeta_{yy} = 0 ; \tag{16}$$

$$\xi_{xx} - 2\zeta_{zx} = 0, \quad \eta_{yy} - 2\zeta_{yz} = 0, \quad \zeta_{zz} - 2\xi_{zx} = 0, \quad \zeta_{zz} - 2\eta_{yz} = 0 ; \tag{17}$$

$$\xi_{xy} - \zeta_{yz} = 0, \quad \eta_{xy} - \zeta_{zx} = 0, \quad \xi_{zx} - \eta_{yz} + \zeta_{zz} = 0. \tag{18}$$

These several sets lead to the following respectively

$$\xi = y\varphi_1(x) + \varphi_2(z, x) \equiv z\varphi_3(x) + \varphi_4(x, y) ; \tag{19}$$

$$\eta = x\varphi_5(y) + \varphi_6(y, z) \equiv z\varphi_7(y) + \varphi_8(x, y) ; \tag{20}$$

$$\zeta = x\varphi_9(z) + \varphi_{10}(y, z) \equiv y\varphi_{11}(z) + \varphi_{12}(z, x) ; \tag{21}$$

$$\xi_x - 2\zeta_z = \psi_1(y, z), \quad \eta_y - 2\zeta_z = \psi_2(z, x), \qquad (22)$$
$$\zeta_z - 2\xi_x = \psi_3(x, y), \quad \zeta_z - 2\eta_y = \psi_4(x, y);$$

$$\xi_x - \zeta_z = \psi_5(z, x), \quad \eta_y - \zeta_z = \psi_6(y, z), \qquad (23)$$
$$\xi_x - \eta_y + \zeta_z = \psi_7(x, y).$$

The equations (22) give

$$-3\xi_x = \psi_1(y, z) + 2\psi_3(x, y), \qquad (24)$$

$$-3\zeta_z = 2\psi_1(y, z) + \psi_3(x, y). \qquad (25)$$

The second form of ξ in (19) and the first form of ζ in (21) give respectively

$$-3\xi_x = -3z\varphi_{3_x}(x) - 3\varphi_{4_x}(x, y), \qquad (26)$$

$$-3\zeta_z = -3x\varphi_{9_z}(z) - 3\varphi_{10_z}(y, z). \qquad (27)$$

Identifying (24) and (26), (25) and (27) respectively, we have

$$-3z\varphi_{3_x}(x) - 3\varphi_{4_x}(x, y) = \psi_1(y, z) + 2\psi_3(x, y), \qquad (28)$$

$$-3x\varphi_{9_z}(z) - 3\varphi_{10_z}(y, z) = 2\psi_1(y, z) + \psi_3(x, y). \qquad (29)$$

Since the left member of (28) is linear in z, $\psi_1(y, z)$ must be linear in z and by (14) simultaneously linear in y with no term of the form yz; hence

$$\psi_1(y, z) \equiv -3\alpha_1 y - 3\alpha_2 z - 3\alpha_3; \qquad (30)$$

similarly from equations (29) and (16)

$$\psi_3(x, y) \equiv -3\beta_1 x - 3\beta_2 y - 3\beta_3. \qquad (31)$$

Hence from (28)

$$\varphi_{3_x}(x) = \alpha_2, \quad \varphi_3(x) = \alpha_2 x + \alpha_4; \qquad (32)$$

$$\varphi_{4_x}(x, y) = (\alpha_1 + 2\beta_2)y + 2\beta_1 x + (\alpha_3 + 2\beta_3), \qquad (33)$$
$$\varphi_4(x, y) = \alpha_5 xy + \alpha_6 x^2 + \alpha_7 x + \alpha_8 y + \alpha_9.$$

Therefore finally, after making a convenient change of constants,

$$\xi(x, y, z) \equiv a + ex + hy + lz + \alpha x^2 + \beta xy + \gamma zx. \qquad (34)$$

In a similar manner η and ζ are found to have the following forms, where the choice of constants is determined by the equations (17) and (18)

$$\eta(x, y, z) \equiv b + fx + jy + mz + \alpha xy + \beta y^2 + \gamma yz, \quad (35)$$

$$\zeta(x, y, z) \equiv c + gx + ky + nz + \alpha zx + \beta yz + \gamma z^2. \quad (36)$$

Then the most general infinitesimal projective transformation of ordinary space has the form

$$Uf \equiv (a + ex + hy + lz + xu)\frac{\partial f}{\partial x} + (b + fx + jy + mz + yu)\frac{\partial f}{\partial y} + (c + gx + ky + nz + zu)\frac{\partial f}{\partial z}, \quad (37)$$

where $$u(x, y, z) = \alpha x + \beta y + \gamma z. \quad (38)$$

The finite forms of the transformation are found by means of Lie's theorem * by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1, z_1)} = \frac{dy_1}{\eta(x_1, y_1, z_1)} = \frac{dz_1}{\zeta(x_1, y_1, z_1)} = d\varepsilon,$$

with the initial conditions

$$x_1 = x, \quad y_1 = y, \quad z_1 = z, \quad \varepsilon = 0;$$

the integration yields

$$x_1 = L / P, \quad y_1 = M / P, \quad z_1 = N / P,$$

where L, M, N, P are of the form

$$\alpha_i x + \beta_i y + \gamma_i z + \delta_i,$$

α_i, β_i, γ_i, δ_i being constants and the index i assuming the values 1, 2, 3, 4 successively.

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* Lie: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1891, Theorem 13, p. 218.