INFINITESIMAL TRANSFORMATIONS OF CONCENTRIC CONICS.

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A FAMILY of curves is invariant under the transformations of a continuous group of transformations when the family is invariant under the infinitesimal transformations which generate the group. A family is invariant under an infinitesimal transformation when the differential equation of the family admits of the infinitesimal transformation.

The criterion that a given differential equation of the mth order in x, y

$$\Phi(x, y, y', y'', \cdots, y^{(m)}) = 0, \tag{1}$$

admit of a known infinitesimal point transformation

$$Uf \equiv \xi(x, y)f\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}, \tag{2}$$

is that

$$U^{(m)}\Phi \equiv 0, \tag{3}$$

where

$$U^{(m)}f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \cdots + \eta^{(m)}\frac{\partial f}{\partial y^{(m)}},$$

$$\eta' \equiv \frac{d\eta}{dx} - y'\frac{d\xi}{dx}, \cdots, \eta^{(m)} \equiv \frac{d\eta^{(m-1)}}{dx} - y^{(m)}\frac{d\xi}{dx},$$

give the mth extension of the original point transformation, Uf.

Conversely, if the differential equation is given and the infinitesimal transformation unknown, the condition (3) may be turned to account to find the forms of those infinitesimal point transformations of which the given differential equation admits. This converse problem is an integration problem not capable of general solution; in fact there are differential equations of the mth order which do not admit of infinitesimal point transformations.

If in particular the equation (1) is of the second order and can be put in the form

$$y'' - \omega(x, y, y') = 0, \tag{4}$$

the criterion becomes

$$\begin{aligned}(\eta_y - 2\xi_x - 3\xi_y y')\,\omega - \xi_{yy}y'^3 + (\eta_{yy} - 2\xi_{xy})y'^2 \\ + (2\eta_{xy} - \xi_{xx})y' + \eta_{xx} \qquad (5) \\ - \{\xi\omega_x + \eta\omega_y + [\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2]\,\omega_{y'}\} \equiv 0\end{aligned}$$

for all values of x, y, y'.

The differential equation of the first order

$$\omega(x, y, y') = 0$$

admits of the transformation (2) if the quantity within the braces in the preceding identity vanishes identically for all values of x, y, y'.*

In this note these well-known criteria are employed to determine those continuous groups of point transformations which leave the family of all concentric conics of the xy-plane invariant. The problem proves to possess an interesting solution.

The differential equation of a family of confocal conics is

$$(xy' - y)(yy' + x) - c^2y' = 0\,;\dagger \qquad (6)$$

from this the second order differential equation of all conics having the same centre is found to be

$$xyy'' + xy'^2 - yy' = 0, \qquad (7)$$

an equation which can be put immediately into the form (4),

$$y'' = \frac{y'}{x} - \frac{y'^2}{y}.$$

If the required infinitesimal transformation be taken in the form (2) the criterion (5) becomes

$$\begin{aligned}x^2y(y\xi_{yy} - \xi_y)y'^3 + x(2xy^2\xi_{xy} + 2y^2\xi_y - xy^2\eta_{yy} - xy\eta_y + x\eta)y'^2 \\ - y(2x^2y\eta_{xy} + 2x^2\eta_x - x^2y\xi_{xx} - xy\xi_x + y\xi)y' \qquad (8) \\ - xy^2(x\eta_{xx} - \eta_x) \equiv 0\,;\end{aligned}$$

* See theorems 21 and 35 of Lie's Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1891.

† This intermediate differential equation is introduced for use in a subsequent note. A neat derivation and integration of it are to be found in Jordan, Cours d'Analyse, 2d edition, vol. 1, No. 167; vol. 3, No. 36. It is obvious that the equation (7) is obtained in simpler manner directly from the equation $lx^2 + my^2 + n^2 = 0$.

whence the following equations of condition for determining the forms of the functions $\xi(x, y)$ and $\eta(x, y)$:

$$y\xi_{yy} - \xi_y = 0, \tag{9}$$

$$2xy^2\xi_{xy} + 2y^2\xi_y - xy^2\eta_{yy} - xy\eta_y + x\eta = 0, \tag{10}$$

$$2x^2y\eta_{xy} + 2x^2\eta_x - x^2y\xi_{xx} - xy\xi_x + y\xi = 0, \tag{11}$$

$$x\eta_{xx} - \eta_x = 0. \tag{12}$$

The equations (9) and (12) demand that

$$\xi \equiv y^2X(x) + \varphi(x), \tag{13}$$

$$\eta \equiv x^2Y(y) + \psi(y); \tag{14}$$

these forms on being introduced into the equations (10) and (11) reduce them respectively to

$$y^3d(4xX) - x^3f(Y) - xf(\psi) = 0, \tag{15}$$

$$x^3d(4yY) - y^3f(X) - yf(\varphi) = 0, \tag{16}$$

where

$$f\{\rho(t)\} \equiv t^2\frac{d^2\rho}{dt^2} + t\frac{d\rho}{dt} - \rho. \tag{17}$$

The equations (15) and (16) give

$$X \equiv \frac{x^3}{16}\frac{f(Y)}{y^3} + \frac{x}{8}\frac{f(\psi)}{y^3}, \tag{18}$$

$$Y \equiv \frac{y^3}{16}\frac{f(X)}{x^3} + \frac{y}{8}\frac{f(\varphi)}{x^3}. \tag{19}$$

Then X is a function of x alone and Y a function of y alone if

$$\frac{f(X)}{x^3} = k, \quad \frac{f(Y)}{y^3} = l, \quad \frac{f(\varphi)}{x^3} = m, \quad \frac{f(\psi)}{y^3} = n, \tag{20}$$

where k, l, m and n are constants.

But it is readily seen that k and l must be zero; for, taking the abridged forms

$$X_1(x) \equiv \frac{x^3}{16}\frac{f[Y_1(y)]}{y^3}, \tag{21}$$

$$Y_1(y) \equiv \frac{y^3}{16}\frac{f[X_1(x)]}{x^3}, \tag{22}$$

we have

$$Y_1(y) \equiv \frac{ky^3}{16}, \quad \frac{f[X_1(x)]}{x^3} \equiv k;$$

whence

$$X_1(x) \equiv \frac{kx^3}{8} + ax + \frac{b}{x}; \tag{23}$$

but

$$f[Y_1(y)] \equiv \frac{k}{2} y^3,$$

which gives

$$X_1(x) \equiv \frac{kx^3}{32}, \tag{24}$$

from (21). These two forms (23) and (24) for $X_1(x)$ are incompatible. In the same way we find inconsistent values for l. We have then incidentally the fact that it is impossible to find two functions $X_1(x)$ and $Y_1(y)$ satisfying the identities (21) and (22).

Hence the functions X, Y, φ, and ψ are found by integrating the following ordinary linear differential equations of the second order:

$$f(X) = 0, \quad f(Y) = 0, \quad f(\varphi) = mx^3, \quad f(\psi) = ny^3;$$

this integration yields

$$X = \alpha x + \frac{\gamma}{x}, \quad \varphi = \frac{m}{8} x^3 + \varkappa x + \frac{\lambda}{x};$$

$$Y = \beta y + \frac{\delta}{y}, \quad \psi = \frac{n}{8} y^3 + \mu y + \frac{\nu}{y};$$

the equations (9)–(12) or their equivalents (13)–(19) impose the following limitations on the constants:

$$\alpha = \beta = \frac{m}{8} = \frac{n}{8}, \quad \gamma = \delta = 0;$$

whence finally

$$X = \alpha x, \quad Y = \alpha y;$$

$$\varphi = \alpha x^3 + \varkappa x + \frac{\lambda}{x}, \quad \psi = \alpha y^3 + \mu y + \frac{\nu}{y};$$

$$\xi = \alpha(xy^2 + x^3) + \varkappa x + \frac{\lambda}{x}, \quad \eta = \alpha(x^2 y + y^3) + \mu y + \frac{\nu}{y}.$$

Hence the most general infinitesimal point transformation which leaves the family of all concentric conics invariant is

$$Uf \equiv \left\{ \alpha x(x^2 + y^2) + \kappa x + \frac{\lambda}{x} \right\} \frac{\partial f}{\partial x}$$

$$+ \left\{ \alpha y(x^2 + y^2) + \mu y + \frac{\nu}{y} \right\} \frac{\partial f}{\partial y}.$$

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A SOLUTION OF THE BIQUADRATIC BY BINOMIAL RESOLVENTS.

BY DR. GEORGE P. STARKWEATHER.

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THE solution of a given equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

consists in making it depend upon a series of resolvent equations

$$R_1 = 0, \quad R_2 = 0, \cdots$$

whose solution may be effected by known methods. Thus the general quadratic is reduced to a binomial $x^2 = a$, the cubic to a quadratic and a binomial cubic, and the biquadratic to a cubic and three quadratic equations. In all the solutions of the biquadratic the writer has seen these resolvent equations are not binomial, although Galois' theory shows us that they may so be taken in an infinite variety of ways, according to the particular system of resolvent functions chosen. In selecting such a system it is of course desirable to find one which will give as simple results as possible, and after some trial the set employed in the following lines seemed to be the best. It is hoped this solution will be of interest from two points of view: 1° as giving a *new* solution of the biquadratic in which the roots are given explicitly, *i. e.*, ready for calculation; 2° as affording an interesting application of Galois' methods.