ticulation is special. Thus for $F = 1$ or 2 it can never happen that there shall be a positive K without a negative one.

Similarly, if $F = 3$ there is no special reticulation for $p = 3s + 1$ while for $p = 3s$ or $3s - 1$ there is always one with a negative K.

When $F = 4$ we can have $K_1 = -p - 3$ belonging to a special reticulation when p is even, while for p odd there is no special reticulation.

When $F = 5$ we finally get exceptional reticulations provided $p = 5s - 1$ or $5s + 2$ and the s is rightly chosen. The simplest is that in Professor White's table, 5_{14}, 14_5.

Again, when $F = 6$ there are sometimes exceptional reticulations for $p = 6s + 3$. The simplest is again one given in Professor White's tables 6_{11}, 11_6.

Other special reticulations occur for $F = 7$. The simplest is 7_{10}, 10_7 for $p = 10$.

In all attempts to realize these exceptional reticulations by construction I have failed. Nor do I see any way of proving that they cannot be constructed. This last once done would show Professor White's method to be exhaustive.

LINCOLN NEB.,
April, 1898.

LIMITATIONS OF GREEK ARITHMETIC.

BY MR. H. E. HAWKES.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

I PROPOSE to discuss in the present paper the number system of the Greeks, and to show how their arithmetical notions were limited by their geometrical symbolism. My argument is based chiefly on Euclid's Elements. This is not a serious limitation, for, firstly, the Elements give us practically all that Greek mathematicians knew on the subject, prior to 300 B. C., and, secondly, little was accomplished in this direction during the following three or four centuries. We may, therefore, consider Euclid's theory of number as representative.

I shall first attempt to show that Euclid naturally expressed and investigated problems relating to number and magnitude through the symbolism of lines. For example, he proves* in Book II, Proposition I, the general distributive law, which in his geometrical symbolism he expresses as follows :

"If there be two straight lines, and one of them be cut into any number of segments, the rectangle contained by the two straight lines equals the rectangles contained by the undivided line and the segments of the divided line." This in our literal symbolism would be expressed thus,

$$a(b + c + d + \cdots) = ab + ac + ad + \cdots$$

Later in the same book,† Proposition XI, Euclid shows how "to cut a given straight line so that the rectangle contained by the line and either segment equals the square on the remaining segment"—a problem which we should express thus—given a quantity a, find x and y such that they satisfy the equations

$$x + y = a, \quad ax = y^2.$$

It is a remarkable fact that the rest of the book is chiefly occupied by proofs of particular cases of the distributive law. On account of his unwieldy symbolism Euclid does not recognize the fact that he has already proved the general case. Though he does not expressly make such a statement, it is very probable that Euclid intended the problems of this book to be true for magnitudes in general, for which the lines are merely the symbols.*

In Book V, Euclid considers the ratio of homogeneous magnitudes, and in the following books he applies the theorems there proven to particular kinds of magnitudes, as surfaces, solids, and numbers.† Throughout Book V, the symbol by which he expresses these general magnitudes is the line. How cumbersome his notation is, can be gathered from the fact that proofs which consume thirty-four pages of Heiberg's edition of the Elements would occupy only three or four expressed in our literal symbolism.

In Book X the problem of incommensurability is investigated, a question which does not appear in the Elements up to that point owing to the power of Euclid's theory of ratio. His investigations in this book on the incommensurability

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* Euclid's Elementa, edited by I. L. Heiberg, Leipzig, 1883, vol. 1, p. 118.

† Heiberg : loc. cit., vol. 1, p. 153.

* Cf. M. Cantor : Vorlesungen über die Geschichte der Mathematik, vol. 1, p. 249 ; also J. Gow: Short history of Greek mathematics, Cambridge, 1884, p. 72.

† Heiberg : loc. cit., vol. 4, p. 138; vol. 4, p. 169; vol. 2, p. 334.

of his magnitude symbols are precisely analogous to the work of Dedekind, Cantor, and Weierstrass on irrationalities. Here he considers all lines of the type

$$A = \sqrt{\sqrt{x} \pm \sqrt{y}}$$

where x and y are commensurable. He finds that there are twenty-five distinct or independent classes of such lines.* He also shows how to find individuals of each class, a process which leads him to the solution of certain equations of the fourth degree. This problem he attacks and solves with his line notation.† It is certainly very wonderful that Euclid could exhaust the subject of incommensurable lines of this type by means of his clumsy notation, but in this very book the weakness of the line symbolism shows itself most clearly in the limitations which it imposed on his concept of incommensurability. He understood the kind of incommensurability which we call square root, for he could construct a square equivalent to a given rectangle whose sides were commensurable. He also arrived at what we know as the fourth root of a quantity by a similar construction. His symbolism would allow him to consider the eighth root, the sixteenth root, etc., but that is as far as his line notation would permit him to go in space as we perceive it. We see that the choice of a magnitude symbol which depends on our space intuitions is a serious limitation.

If we ask how far the Greeks went toward arranging their line symbols into a dense multiplicity or a continuous system, we must answer that they took no steps at all. Euclid assumes that the sum of two lines is a line, that a short line taken from a long line leaves a line. The difficulty which arises if we wish to take a long line from a short one never seems to trouble him. The product of two lines is a rectangle, of three lines a solid. He does not concern himself with the process of division. He considers incommensurable lines, as we have seen, but takes no steps toward arranging his symbols in order of magnitude much less as a dense multiplicity or a continuum.

Let us now consider the effect of the line notation on the development of the number system of the Greeks. It will be evident that Euclid did not consider the line as merely a con-

* See De Morgan's article in the Penny Encyclopedia: "Irrationality."

† S. A. Christensen: "Über Gleichungen vierten Grades im zehnten Buch der Elemente Euclid's." *Zeitschrift für Mathematik und Physik, Historisch-literarische Abteilung*, 1889, p. 201.

venient graphical representation for number, but that he regarded the line as the fundamental symbol for any magnitude of which number was a special case. I consider in this paper only the theoretical arithmetic or *Arithmetica*, not the practical arithmetic or *Logistica* of the shopkeeper or the surveyor. The two were entirely distinct in the mind of the Greek.

We cannot find a time when the Greeks were not familiar with the smaller positive integers and did not have names for them. One would say that these would be the most natural symbols for mathematical magnitudes. It may be, however,* that the discovery of the incommensurable by Pythagoras destroyed the confidence of the Greeks in the integers as magnitude symbols. The finding of a line which could not be expressed in terms of a given line by means of integers may well have suggested to them that the line would be capable of expressing more magnitudes than the integers and thus give rise to a richer multiplicity. Thus they were led to leave reasoning by means of numerical notation to the shopkeeper and practical mathematician. The pure mathematician considered the line as the only safe notation. The separation of geometry from analysis was as complete as Weierstrass could wish, but with the opposite order of importance.

Consider now the Greek conception of operations on integers and note the constantly narrowing effect of the line notation.

Euclid assumes that two numbers added make a number† as is evident from the fact that a line plus a line generates a new line. He also assumes that a number taken from a number leaves a number.‡ He infers that we always take the less from the greater and never does he betray a suspicion that a case may arise where the greater might be subtracted from the less. When he solves equations by his line notation§ he finds only positive roots. Even Diophantos (250 ? A.D.)|| rejects negative solutions to his problems, and nowhere to the end of Greek mathematics do we find a negative number.¶ It is easy to see that taking a long line from a short one would seem so absurd that they would never think of the operation of taking a greater number

* H. B. Fine : Number System of Algebra, Boston, 1890, p. 97.

†Heiberg : loc. cit., vol. 2, p. 393 ff.

‡ Vol. 2, p. 395.

§ Vol. 1, p. 153.

|| T. L. Heath : Diophantos of Alexandria, Cambridge, 1885, pp. 82, 201, 207.

¶ M. Cantor : Geschichte der Mathematik, vol. 1, p. 401.

from a less one, since from their point of view the latter would be a special case of the former. Thus the line symbolism limited the Greek number system by one-half in depriving it of negative numbers.

From Plato* we learn that a number made up of the product of two unequal factors is an oblong number; if the factors are equal it is a square number. Similarly they had solid numbers and cube numbers. Suppose however they wished to express a product composed of four factors. If the numbers were to represent lines, the product would carry them into four dimensions. This was a difficulty which Euclid saw and could not meet. For whenever he considered the product of two numbers he first postulated the existence of that product.† For instance, Book IX, Proposition XXVIII: "If an odd number multiplying an even number *make any number*, the product will be even." Again Book IX, Proposition I: "If two similar plane numbers multiplying each other *make any number*, the number produced by them will be a square." Plainly he was not sure of the existence of the product, if the numbers were such as would involve more than three dimensions for their geometrical expression.

Thus the line notation imposes a further bond on the Greek number system in that it makes the consideration of more than three factors a peculiarly delicate matter. This very meager number system which we have sketched, viz., the positive integers which can be considered as the product of not more than three factors, is the end of the growth of the Greek number system until the time of Diophantos who introduced the rational fraction.‡ "No case of simple division occurs in Greek arithmetical literature."§ The fraction was to the Greeks the ratio of two numbers and did *not* generate a new number. Euclid in Book V. considers the ratio of homogeneous quantities, but he does not conceive of these ratios as quantities at all. They were not subject to the axioms to which all quantities are subject, *e. g.*, in Book V, Proposition XI,‖ he proves, "Ratios which are equal to the same ratio are equal to each other." It is impossible to tell how far the fact that division does not occur as an explicit process is due to the lack of any well-defined similar process on lines; but, judging from the

* Plato: Theaetetus, translated by Jowett, vol. 3, p. 347.
† Heiberg: loc. cit., vol. 2, pp. 340–347, 396 ff.
‡ M. Cantor: Geschichte der Mathematik, vol. 1, pp. 159, 405.
§ J. Gow: Short history of Greek mathematics, p. 51.
‖ Heiberg: loc. cit., vol. 2, p. 35.

close dependence of the operations of subtraction and multiplication of numbers on the same operations for lines, it would seem that there must be some connection.

It is evident that the Greeks were even farther from a continuum of numbers than they were from a continuum of their line symbols. For no number in their system existed which expressed their incommensurable lines, not to mention the countless kinds of incommensurability of which they knew nothing.

These considerations indicate that Greek mathematics rested on a very narrow basis so long as it clung to its line notation. The sense of rigor, as shown by postulating the existence of a product of two factors certainly would not allow them to assume a continuous system, as less careful mathematicians have done. This line notation did not admit of sufficient expansion to allow them to establish on *that* such a system. Thus, until the foundation of their mathematical science was utterly changed, an advance to algebra and calculus was impossible.

YALE UNIVERSITY,
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MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.

BY PROFESSOR JAMES PIERPONT.

IN treating the theory of maxima and minima in my lectures this year I have been astonished to find that the presentation of this theory in all English and American textbooks on the calculus which I could consult was false. That the older editions of such standard treatises as Todhunter, Williamson, and Byerly should be wrong in this particular was not astonishing since it was only in 1884 that Peano in his critical notes to the Calcolo Differenziale of Genocchi called attention to the point in question. Since then L. Scheeffers,* O. Stolz,† and von Dantscher‡ have devoted memoirs to this interesting but difficult subject and their results have found a place in the new edition of the Cours d'Analyse of C. Jordan and the Grundzüge of O. Stolz.

* *Mathematische Annalen*, vol. 35, p. 541.
† *Sitzungsberichte Vienna Academy*, 1890 (June).
‡ *Annalen*, vol. 42, p. 89.