PERIODIC DEVELOPMENTS.

\[ F(z) = \sum_{n=0}^{\infty} a^n z^n, \quad (|a| < 1), \]

is single-valued, provided \(|a|\) is not too large.

The proof is as follows. Evidently

\[
\left| \frac{f(z) - f(z')}{z - z'} \right| = \left| 1 + \sum_{n=1}^{\infty} \frac{z^n + z' n + \cdots + z'^{n+1}}{(a^n + 1)(a^n + 2)} \right|
\]

\[
\geq 1 - \sum_{n=1}^{\infty} \frac{|z|^{n+1} + |z'|^{n+1}}{(a^n + 1)(a^n + 2)}
\]

\[
\geq 1 - \frac{1}{a + 1} - \frac{1}{a(a - 1)} > 0.
\]

Hence \(|f(z) - f(z')| > 0\), q. e. d.

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BY PROFESSOR ALEXANDER S. CHESSIN.

If we put with Professor S. Newcomb*

\[
E = ev_1 + e^2 v_2 + e^3 v_3 + \cdots
\]

\[
\rho = \log a = e\rho_1 + e^2 \rho_2 + e^3 \rho_3 + \cdots
\]

where \(E\) stands for the equation of the center and \(\rho = \log r\), then \(v_i\) and \(\rho_i\) will be of the form

\[
i v_i = \frac{1}{2} \sum k_j^{(0)} \sin j\zeta,
\]

\[
i \rho_i = \frac{1}{2} \sum h_j^{(0)} \cos j\zeta,
\]

\((j = i, \ i - 2, \ i - 4, \ \cdots, \ -i)\).

the coefficients $k_j^{(i)}$ and $h_j^{(i)}$ being rational numerical fractions subject to the conditions

$$ k_j^{(i)} = - k_{-j}^{(i)}; \quad h_j^{(i)} = h_{-j}^{(i)}. $$

We propose to give in this note formulas by which these coefficients can be computed for any value of $i$ and $j$.

If we put

$$ (5) \quad E = \sum_{i=1}^{\infty} H_i \sin i \zeta, $$

$$ (6) \quad \rho - \log a = \frac{1}{2} A_0 + \sum_{i=1}^{\infty} A_i \cos i \zeta, $$

then the comparison with formulas (1)-(4) gives

$$ (7) \quad H_i = \sum_{m=0}^{\infty} \frac{k_i^{(i+2m)}}{i + 2m} e^{i+2m}, $$

$$ (8) \quad A_i = \sum_{m=0}^{\infty} \frac{h_i^{(i+2m)}}{i + 2m} e^{i+2m}. $$

On the other hand it can be shown* that

$$ (9) \quad H_i = \frac{2\sqrt{1 - e^z}}{z} \sum_{j=0}^{\infty} \frac{j^z}{q!} \left( \frac{e}{2} \right)^{j+z} N_{-i,j,q}, $$

where $j$ and $q$ assume all integral positive values (zero included) such that

$$ j + q = i, \quad i + 2, \quad i + 4, \ldots $$

If we develop $\sqrt{1 - e^z}$ and put

$$ (10) \quad H_i^{(2m)} = \sum_{j=0}^{\infty} \frac{j^z}{q!} N_{-i,j,q} \quad (i + j + q = 2m), $$

then formula (9) becomes

$$ (11) \quad H_i = \frac{2}{z} \left( \sum_{m=0}^{\infty} \left( \frac{e}{2} \right)^{2m} \frac{H_i^{(2m)} - 2H_i^{(2m-2)} - \ldots - 1.3 \cdots (2m-3) 2^m H_i^{(2m-3)}}{m!} \right). $$

Comparing this formula with (7) we conclude that

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\[ k_i^{(i+2m)} = \left( \frac{i+2m}{i} \right) \cdot \frac{1}{2^{i+2m-1}} \left[ H_i^{(2m)} - 2H_i^{(2m-2)} \right. \]

\[ - \frac{1}{2} 2^i H_i^{(2m-4)} - \cdots - \frac{1.3 \cdots (2m-3)}{m!} 2^m H_i^0 \].

By this formula the computation of the coefficients \( k_i^{(i)} \) is reduced to the computation of Cauchy’s numbers for which the author has given a general formula.*

In order to obtain a similar expression for the coefficients \( h_j^{(i)} \) we must first derive a development in powers of the eccentricity for the coefficients \( A_i \). To this end we remark that

\[ \frac{dp}{de} = \frac{d \log r}{de} = \frac{dr}{rde} = \frac{1}{e} \left( \frac{a}{r} \right) - \frac{1 - e^2}{e} \left( \frac{a}{r} \right)^2. \]

On the other hand we have \( \dagger \)

\[ \frac{a}{r} = 1 + 2 \sum_{i=1}^{i=m} J_i(ie) \cos i \zeta. \]

\[ \left( \frac{a}{r} \right)^2 = \frac{1}{\sqrt{1 - e^2}} + \sum_{i=1}^{i=m} G_i^{(2)} \cos i \zeta \]

where \( J_i(ie) \) is a Bessel’s function and

\[ G_i^{(0)} = 2 \sum_j \sum_q \left( \frac{e}{q} \right)^{j+q} N_{i,j,q} (j + q = i, i + 2, i + 4, \cdots). \]

Hence we may write that

\[ \frac{dp}{de} = \frac{1 - \sqrt{1 - e^2}}{e} + \frac{1}{e} \sum_{i=1}^{i=m} \left[ 2J_i(ie) - (1 - e^2)G_i^{(0)} \right] \cos i \zeta. \]

Now, it follows from (6) and (8) that

\[ e \frac{dp}{de} = \frac{1}{2} e \frac{dA_i}{de} + \sum_{i=1}^{i=m} \sum_{m=0}^{m=\infty} e^{i+2m} h_i^{(i+2m)} \cos i \zeta \]

which, compared with the preceding formula, shows that

\[ 2J_i(ie) - (1 - e^2)G_i^{(0)} = \sum_{m=0}^{m=\infty} e^{i+2m} h_i^{(i+2m)} \]

and we only need to find the coefficient of \( e^{i+2m} \) in the left hand side to obtain the required expression for \( h_i^{(i+2m)}. \)

From (9) and (13) follows that

\[
(1 - e^\phi) G_i^{(i)} = i \sqrt{i - e^\phi} H_i
\]

\[
= 2 \left( \frac{\phi}{2} \right) ^i (1 - e^\phi) \sum_{m=0}^{\infty} H_i^{(2m)} \left( \frac{\phi}{2} \right) ^{2m}
\]

\[
= 2 \left( \frac{\phi}{2} \right) ^i \sum_{m=0}^{\infty} [H_i^{(2m)} - 4H_i^{(2m-2)}] \left( \frac{\phi}{2} \right) ^{2m}
\]

so that the coefficient of \( e^{i+2m} \) in \((1 - e^\phi) G_i^{(i)}\) is found to be

\[
\frac{1}{2^{i+2m-1}} \left[ H_i^{(2m)} - 4H_i^{(2m-2)} \right]
\]

while the coefficient of the same power of \( e \) in \( 2J_i(i\phi) \) is

\[
(-1)^m \frac{1}{2^{i+2m-1}} \cdot \frac{i^{i+2m}}{m! (i + m)!}.
\]

Hence, we conclude that

\[
(14) \quad H_i^{(i+2m)} = \frac{1}{2^{i+2m-1}} \left[ 4H_i^{(2m-1)} - H_i^{(2m)} + \frac{(-1)^m}{m! (i + m)!} \right]
\]

which is the desired expression for the coefficients \( h_i^{(0)} \).

To conclude we will express the coefficients \( h_i^{(0)} \) by means of the \( k_i^{(0)} \). To this end we multiply formula (7) by \( \sqrt{1 - e^\phi} \) and develop the right hand side in powers of \( e \). Thus we obtain

\[
\sqrt{1 - e^\phi} H_i = \sum_{m=0}^{\infty} e^{i+2m} \left[ \frac{k_i^{(i+2m)}}{i + 2m} - \frac{1}{2} \frac{k_i^{(i+2m-2)}}{i + 2m - 2} - \cdots \right]
\]

\[
\cdots - \frac{1.3 \cdots (2m - 3)}{m!} \left( \frac{1}{2} \right) ^m \frac{k_i^i}{i}
\]

d, therefore,

\[
H_i^{(i+2m)} = \frac{2(-1)^m \left( \frac{i}{2} \right) ^{i+2m}}{m!(i + m)!} - \frac{i k_i^{(i+2m)}}{i + 2m} + \frac{1}{2} \frac{i k_i^{(i+2m-2)}}{i + 2m - 2}
\]

\[
+ \frac{1}{2!} \left( \frac{1}{2} \right) ^i \frac{i k_i^{(i+2m-4)}}{i + 2m - 4} \cdots + \frac{1.3 \cdots (2m - 3)}{m!} \left( \frac{1}{2} \right) ^m \frac{i k_i^i}{i}
\]

which formula enables us to compute the values of the \( h_i^{(0)} \) directly from the \( k_i^{(0)} \).

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