Further, formula (9) of § 11 becomes for $q = m - 2$

$$\begin{vmatrix} i_1 i_2 \cdots i_{m-2} \\ j_1 i_2 \cdots j_{m-2} \end{vmatrix}_A = D^{m-3} \begin{vmatrix} i_{m-1} i_m \\ j_{m-1} j_m \end{vmatrix}_a.$$

Hence the transformation (12) takes the form*

$$(12_1) \qquad W'_{i_{m-1} i_m} = D^{\frac{m-4}{2}} \sum_{j_{m-1} j_m}^{1,\cdots,m} \begin{vmatrix} i_{m-1} i_m \\ j_{m-1} j_m \end{vmatrix}_a W_{j_{m-1} j_m}.$$

18. We may enunciate the results proven in §§ 16–17 for the individual transformations of the groups concerned :

To any given transformation (a_{ij}) *of determinant* D *of the general m-ary linear homogeneous group* G_m, *there corresponds a transformation* $[a]_{m-2}$ *of the* $(m-2)^d$ *compound* $C_{m, m-2}$ *which gives rise to a linear transformation upon its system of Pfaffian invariants, viz :*

1° : *for m odd, the m-ary transformation,*

$$\overline{F}'_i = D^{\frac{m-3}{2}} \sum_{j=1}^{m} a_{ij} \overline{F}_j \qquad (i = 1, \cdots, m),$$

which for $D = 1$, *is precisely the given transformation of* G_m.

2°. *for m even, the* $\frac{1}{2}m(m-1)$ *-ary transformation* (12) *or* (12_1), *where, for* $D = 1$, (12_1) *belongs to the second compound of* G_m, *and* (12) *to the* $(m-2)^d$ *compound of the* $(m-1)^{st}$ *compound of* G_m.

University of California,
August 9, 1898.

A SECOND LOCUS CONNECTED WITH A SYSTEM OF COAXIAL CIRCLES.

BY PROFESSOR THOMAS F. HOLGATE.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, August 19, 1898.)

In a paper read before this Society at its Toronto Meeting and published in the Bulletin for November, 1897, I

* We may verify (12_1) directly, using the method of § 6 for $q = 2$. The presence of the factor $D^{\frac{m-4}{2}}$ influences only the transformations $A_{kk'}$. There occurs, however, some difficulty as to signs in passing from the W's to the F's. Likewise the results of §§ 11–14 could doubtless be proved by the method of § 6.

presented the locus of the common point of three concurrent lines which are tangent to three circles of a coaxial system, each line tangent to two of the three circles and each circle touched by two of the three lines, the circles and lines thus forming a continuous chain, which for a proper or non-degenerate circuit of the kind described proved to be a pair of parabolas having the line of centers of the coaxial system for a common directrix and the two finite points of intersection of the circles for foci.

For simplicity in performing the necessary eliminations in the former paper one of the chosen lines was drawn parallel to the line of centers, but if this first line be selected at random the eliminations do not prove so difficult as they at first appeared, and, the process being of a more general character, other interesting features of the problem come to the surface. In this paper I propose to present these features and then proceed to the case of period four, *i. e.*, to the case in which the circuit closes with four lines and four circles instead of three.

If now we choose the equation for any one of the circles of the coaxial system, as before, in the form

$$x^2 + y^2 - 2k_\nu x - \delta^2 = 0$$

and the equation of a line through the point (a, b) in the form

$$(x - a) + l_\nu(y - b) = 0,$$

where k_ν and l_ν are parameters determining the particular circle and line under consideration, we find the conditions of contact to be

$$(1)\quad k_1^2 l_1^2 + (\delta^2 - b^2)\, l_1^2 + 2bk_1 l_1 + 2ak_1 - 2abl_1 + (\delta^2 - a^2) = 0,$$

$$(2)\quad k_1^2 l_2^2 + (\delta^2 - b^2)\, l_2^2 + 2bk_1 l_2 + 2ak_1 - 2abl_2 + (\delta^2 - a^2) = 0,$$

$$(3)\quad k_2^2 l_2^2 + (\delta^2 - b^2)\, l_2^2 + 2bk_2 l_2 + 2ak_2 - 2abl_2 + (\delta^2 - a^2) = 0,$$

$$(4)\quad k_2^2 l_3^2 + (\delta^2 - b^2)\, l_3^2 + 2bk_2 l_3 + 2ak_2 - 2abl_3 + (\delta^2 - a^2) = 0,$$

$$(5)\quad k_3^2 l_3^2 + (\delta^2 - b^2)\, l_3^2 + 2bk_3 l_3 + 2ak_3 - 2abl_3 + (\delta^2 - a^2) = 0,$$

$$(6)\quad k_3^2 l_1^2 + (\delta^2 - b^2)\, l_1^2 + 2bk_3 l_1 + 2ak_3 - 2abl_1 + (\delta^2 - a^2) = 0,$$

a set of equations involving the coördinates (a, b) of the point of concurrence and the parameters l_1, l_2, l_3 of the three lines and k_1, k_2, k_3 of the three circles.

From these six equations it would clearly be possible to eliminate a, b, l_1, l_2, l_3, or a, b, k_1, k_2, k_3, and thus obtain a relation among the k's, or among the l's, which would render the circuit possible. Proceeding, however, to find the locus of the point of concurrence for which the circuit is possible, we shall eliminate k_1 from equations (1) and (2), k_2 from equations (3) and (4), and k_3 from equations (5) and (6). These eliminations yield the following three expressions which show the two to two relations among the l's, the geometric significance of which will be clear from a diagram, viz.:

(A) $pl_1^2l_2^2 + ql_1l_2(l_1 + l_2) + r(l_1^2 + l_2^2) + sl_1l_2 + t(l_1 + l_2) = 0,$

(B) $pl_2^2l_3^2 + ql_2l_3(l_2 + l_3) + r(l_2^2 + l_3^2) + sl_2l_3 + t(l_2 + l_3) = 0,$

(C) $pl_3^2l_1^2 + ql_3l_1(l_3 + l_1) + r(l_3^2 + l_1^2) + sl_3l_1 + t(l_3 + l_1) = 0,$

where p is written instead of $4b^2(\delta^2 + a^2 - b^2)$,

q, instead of $4ab(\delta^2 + a^2 - 2b^2)$,

r, instead of $(\delta^4 + 2\delta^2a^2 + a^4 - 4a^2b^2)$,

s, instead of $2(\delta^4 + 2\delta^2a^2 + a^4 - 2\delta^2b^2 - 6a^2b^2)$,

t, instead of $-4ba(\delta^2 + a^2)$.

If l_1 be given any fixed value, equation (A) yields the same two roots for l_2 as equation (C) yields for l_3, hence one of these roots must be the value of l_2, the other the value of l_3, consistent with the chosen value of l_1, as is geometrically evident; so that

$$l_2 + l_3 = -\frac{ql_1^2 + sl_1 + t}{pl_1^2 + ql_1 + r},$$

$$l_2l_3 = \frac{rl_1^2 + tl_1}{pl_1^2 + ql_1 + r}.$$

Substituting these expressions in equation (B) and simplifying we obtain the relation

(P) $(rs - qt - r^2)\,[pl_1^4 + 2ql_1^3 + (2r + s)l_1^2 + 2tl_1] = 0.$

If now the coördinates of the point of concurrence satisfy the relation

$$rs - qt - r^2 = 0,$$

then no matter how the line l_1 is drawn, the proposed system is possible.

Replacing q, r, s, t in this last expression by the functions of a and b which they symbolize, and writing x and y for variables, we obtain for the locus of the point of concurrence the equation

$$(\delta^4 + 2\delta^2x^2 + x^4 + 4x^2y^2)\,(\delta^2 + x^2 - 2\delta y)\,(\delta^2 + x^2 + 2\delta y) = 0.$$

The curve represented by the first factor of this equation is wholly imaginary, while those represented by the second and third factors are parabolas, each having the line of centers of the circles for directrix and a finite point of intersection of the circles for focus as was deduced in the former paper. If the circles of the coaxial system have imaginary intersections, *i. e.*, if δ^2 be negative and hence δ imaginary, the locus becomes wholly imaginary, except in two points, viz., the two limiting points of the system of circles. To make the geometric necessity for this evident we need only to recall that of three circles of such a system one must lie wholly within another.

The conditions necessary for a complete circuit will however be satisfied for any point (a, b) of the plane if some one of the three tangent lines be so chosen as to make the second factor in equation (P) vanish, that is, so as to make

$$pl_1^4 + 2ql_1^3 + (2r + s)l^2 + 2tl = 0.$$

In this case the parameter l of the line must equal zero, or

$$\frac{-a \pm i\delta}{b}, \text{ or } \frac{2ab}{\delta^2 + a^2 - b^2},$$

the first three of these values locating the line l through one or other of the three vertices of the common self-conjugate triangle of the coaxial system, the fourth locating it as the tangent to that circle of the system which passes through the point a, b.

This second factor is in fact simply the form which equation (A), or (B), or (C) would take if the two l's involved were equal, and so if l_3, say, should assume any of the three values which would direct the corresponding line through a vertex of the self-conjugate triangle, as for example, the value zero, then in equation (B) one value of l_2 would also be zero while the other, *i. e.*, l_1, would be of the form

$$\frac{4ab(\delta^2 + a^2)}{\delta^4 + 2\delta^2a^2 + a^4 - 4a^2b^2}.$$

These values for the three l's would give for the k's the values

$$k_2 = \infty , \quad k_1 = k_3 = \frac{a^2 - \delta^2}{2a}.$$

If on the other hand, l_3, should assume the value

$$\frac{2ab}{\delta^2 + a^2 - b^2}$$

which locates the line as tangent to that circle of the system passing through a, b, then l_2 would equal l_3, and l_1 would assume a definite value different from them. The corresponding values for the k's would be

$$k_1 = k_3 = -\frac{\delta^4 + 2\delta^2a^2 + a^4 - \delta^2b^2 + a^2b^2}{2ab^2}, \quad k_2 = \frac{\delta^2 - a^2 - b^2}{-2a}.$$

Hence in either of these cases we obtain a degenerate or improper circuit of period three in which two of the lines involved coincide as do also two of the circles, the lines and circles following each other in the order indicated by the succession of parameters l_1, k_1, l_2, k_2, l_3, k_3, returning to the line whose parameter is l_1, and so closing the circuit.

Thus for any point of the plane as point of concurrence there are four distinct circuits of period three while for any point of our locus we have not only these four improper circuits, but also an infinite number of proper circuits.

Turning now to the case in which the circuit closes with four lines and four circles, we have, instead of the six equations of contact, the following eight, viz.:

$$f(k_1, l_1) = 0, \quad f(k_1, l_2) = 0, \quad f(k_2, l_2) = 0, \quad f(k_2, l_3) = 0,$$
$$f(k_3, l_3) = 0, \quad f(k_3, l_4) = 0, \quad f(k_4, l_4) = 0, \quad f(k_4, l_1) = 0,$$

where $f(k_\mu, l_\nu)$ is as before of the form

$$k_\mu^2 l_\nu^2 + (\delta^2 - b^2)\, l_\nu^2 + 2bk_\mu l_\nu + 2ak_\mu - 2abl_\nu + (\delta^2 - a^2).$$

Using the same abbreviations as for period three and eliminating the k's from the eight conditions of contact, two and two, we obtain

$$\text{(A)} \quad pl_1^2l_2^2 + ql_1l_2\,(l_1 + l_2) + r(l_1^2 + l_2^2) + sl_1l_2 + t(l_1 + l_2) = 0,$$
$$\text{(B)} \quad pl_2^2l_3^2 + ql_2l_3\,(l_2 + l_3) + r(l_2^2 + l_3^2) + sl_2l_3 + t(l_2 + l_3) = 0,$$
$$\text{(C)} \quad pl_3^2l_4^2 + ql_3l_4(l_3 + l_4) + r(l_3^2 + l_4^2) + sl_3l_4 + t(l_3 + l_4) = 0,$$
$$\text{(D)} \quad pl_4^2l_1^2 + ql_4l_1\,(l_4 + l_1) + r(l_4^2 + l_1^2) + sl_4l_1 + t(l_4 + l_1) = 0,$$

If in equation (A) we should give l_1 any particular value and solve for l_2 we should obtain the same values as would be obtained for l_4 from equation (D) when the same value is assumed for l_1. But these are merely the values of l_2 or l_4 for which lines are tangent to the same circles of the coaxial system as is the line l_1, that is, one of these roots is l_2 and the other is l_4.

Hence

$$l_2 + l_4 = -\frac{ql_1^2 + sl_1 + t}{pl_1^2 + ql_1 + r},$$

$$l_2 l_4 = \frac{rl_1^2 + tl_1}{pl_1^2 + ql_1 + r}.$$

Similar considerations will yield from equations (B) and (C) the relations

$$l_2 + l_4 = -\frac{ql_3^2 + sl_3 + t}{pl_3^2 + ql_3 + r},$$

$$l_2 l_4 = \frac{rl_3^2 + tl_3}{pl_3^2 + ql_3 + r}.$$

Equating these two values for $l_2 + l_4$ and for $l_2 l_4$ we obtain, in case $l_1 \neq l_3$, the two relations

$$\text{(Q)}\quad (q^2 - ps)l_1l_3 + (qr - pt)(l_1 + l_3) + (rs - qt) = 0$$

$$\text{(R)}\quad (qr - pt)\, l_1 l_3 + r^2 (l_1 + l_3) + rt = 0.$$

If the equations (A), (B), (C), (D) had been taken in pairs (A), (B) and (C), (D), and given a similar treatment we should have obtained relations between l_2 and l_4 identical with those between l_1 and l_3. In fact, it is easily seen that either pair of lines l_1, l_3, or l_2, l_4, is sufficient to determine the whole circuit, since either pair determines all four circles of the circuit, the circles touched by one line of the pair being different from those touched by the other line of the pair.

Since equations (Q) and (R) are to be satisfied for all values of l_1 and all consistent values of l_3, then must

$$\frac{q^2 - ps}{qr - pt} = \frac{qr - pt}{r^2} = \frac{rs - qt}{rt};$$

and these relations are satisfied in case

$$pt^2 - 2\,qrt + r^2s = 0,$$

which gives us the desired locus.

Replacing p, q, r, s, t again by the functions of a and b which they represent and writing x and y for variables we have for the equation of the point of concurrence in a proper circuit of period four

$$(\delta^8 + 4\delta^6x^2 + 6\delta^4x^4 + 4\delta^2x^6 + x^8 + 16\delta^2x^2y^4)(\delta^4 + 2\delta^2x^2 + x^4 - 2\delta^2y^2 + 2x^2y^2) = 0.$$

The first factor of this locus represents a curve which is wholly imaginary when the circles of the coaxial system have real intersections, *i. e.*, when δ^2 is positive, and which breaks up into two imaginary branches, each of which has real double points at the limiting points of the system when δ^2 is negative.

In case the coaxial system has real intersections, the second factor of the locus represents a real curve of the fourth order which is symmetrically situated with respect both to the line of centers and to the radical axis. It cuts the line of centers only in imaginary points and crosses the radical axis orthogonally at the points $0, \pm \frac{\delta}{\sqrt{2}}$. The curve has an ordinary double point at infinity on the radical axis, the asymptotes being $x = \pm \delta$, between which lines the curve is wholly confined, and two imaginary double points at $\pm i\delta$, 0. If the coaxial system has imaginary intersections, the curve represented by this factor becomes also imaginary except for two real double points at the limiting points of the system.

If in equation (A) we impose the condition that the values of l_2 for any given l_1 shall be equal, *i. e.*, that l_2 shall equal l_4, and hence that the circuit shall degenerate, we find that

$$(q^2 - 4pr)l_1^4 + 2(qs - 2pt - 2qr)l_1^3 + (s^2 - 2qt - 4r^2)l_1^2 + 2(st - 2rt)l_1 + t^2 = 0,$$

or substituting the proper values for p, q, r, s, t, this becomes

$$16b^2(\delta^2 + a^2)[(\delta - b)l_1 - a][(\delta + b)l_1 + a][l_1 + i][l_1 - i] = 0.$$

Hence if the point of concurrence of the lines of the circuit be anywhere on the line $y = 0$ or on either of the lines $\delta^2 + x^2 = 0$, or if for any point of the plane, l_1 be so chosen as to equal

$$\frac{a}{\delta - b}, \quad \frac{-a}{\delta + b}, \; -i, \text{ or } i,$$

that is to say, if for any point of the plane one of the lines of the circuit passes through either of the finite intersection points of the circles or through either of the circular points at infinity (through any of the four common points of the circles of the system) the circuit will degenerate.

It is readily seen from an examination of the geometrical figure that such a degenerate circuit is possible for any point of the plane. If from any point P a straight line be drawn to one of the points of intersection of the coaxial system, the two circles of the system tangent to this line coincide. The second tangent through P to this circle determines a second circle which in turn determines a third line, and so on. If now we start with this last line, designating it l_1, and retrace our steps through line and circle alternately, l_1, k_1, l_2, k_2, l_3, k_3, l_4, k_4, we shall find that our circuit closes, or that we return to the starting line l_1, with an even number of lines and of circles, also that in the case of period four l_2 coincides with l_4 while l_1 and l_3 are distinct and that k_1 coincides with k_4 and k_2 with k_3. It will of course be remembered that a cyclic permutation of parameters in no way affects the circuit and that for period four the parameters whose subscripts are *one* and *three* or *two* and *four* are interchangeable as are also the pairs *one*, *three* and *two*, *four*.

The whole matter of degenerate or improper circuits may be summarized as follows:

If for any point of the plane, and starting with any line through it, we traverse a chain of lines and circles of the kind under consideration, and in our progress come upon a line passing through one of the intersection points of the system of circles, then the chain will return upon itself and will close with an even number of lines and of circles. If we recall the treatment of the degenerate circuits of order three it will be readily seen that:

Starting with any line, if we traverse a chain of the kind under consideration and in our progress come upon a line passing through a vertex of the self-conjugate triangle of the system of circles or upon the tangent to the circle through the point of concurrence, then the chain will return upon itself and will close with an odd number of lines and of circles.

For any point of the plane then there will always be four (degenerate) circuits of any odd period and four of an even period. If for a point there should be a fifth circuit of any period then there will be an infinite number of circuits of the same period. That such points exist for any particular period appears at once from a study of the number of conditions of contact and the number of parameters involved in them. The process here employed is adequate to produce the locus of points which admit circuits of any period, but for periods higher than four the eliminations become exceedingly complicated.

Many interesting phases of this problem appear by making certain transformations of the plane. For instance, a projection of the plane will convert our configuration into the more general one consisting of lines through a point and an equal number of conics through four points, each line tangent to two conics, and each conic touched by two lines. An inversion would convert the lines of the circuit into circles of a coaxial system, leaving the circles of the circuit still circles of a coaxial system. Thus our configuration would come to consist of a given number of circles of one coaxial system and an equal number of a second coaxial system, each circle of either set touching two of the other set, the whole forming a continuous chain. Our locus would then become the locus of one of the four intersection points of the two systems of circles, which moves, the other three remaining fixed, so as always to make such a chain of given period possible.

NORTHWESTERN UNIVERSITY,
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RECIPROCAL TRANSFORMATIONS OF PROJECTIVE COÖRDINATES AND THE THEOREMS OF CEVA AND MENELAOS.

BY PROFESSOR ARNOLD EMCH.

1. Among the great number of correspondences between certain configurations of the plane and space it is interesting and valuable to consider relations of the triangle in connection with certain surfaces. It will be seen that propositions of plane geometry interpreted in Cartesian space lead to geometrical questions of a more general character. In this paper we shall confine ourselves to the theorems of