*windschief* (p. 26) instead of gauche? the two words have practically the same signification. The word *Schein*, so significant in the German, is apparently untranslatable; Professor Holgate adopts projector as the equivalent; this answers sufficiently well in Reye, but misses the point in von Staudt's use of the word (Geometrie der Lage, § 3, p. 12), where the reference is explicitly to the *visual* foundation of the geometry. The translator of von Staudt will be hard put to it to render the term adequately.

The book is clearly and accurately printed, but is spoilt for pleasant handling by its most unusual weight.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR,
*September*, 1898.

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## BURKHARDT'S THEORY OF FUNCTIONS.

*Funktionentheoretische Vorlesungen.* Von HEINRICH BURKHARDT. Erster Teil: *Einführung in die Theorie der analytischen Functionen einer complexen Veränderlichen.* Leipzig, Veit & Co., 1897. 8vo, xii + 213 pp.

THE object of the author in writing the little volume before us has been to furnish an introduction to the theory of functions which is not confined to the presentation of the methods of any one school (Cauchy, Weierstrass, Riemann) but blends these methods as far as possible into an organic whole. The author has been very successful in making his book an introduction not merely to those parts of the theory which have long been classical (algebraic, elliptic, and Abelian functions) but also to the many other important developments of the last thirty years.* The mathematical public may well congratulate itself that a mathematician so thoroughly familiar with all sides of the subject as is Professor Burkhardt has undertaken the task of writing an elementary work along these lines.

We will briefly indicate the subjects treated.

Chapter I is an excellent presentation of the elementary theory of complex numbers and their geometric representation, in which the author has wisely restricted himself to the ordinary complex numbers $a + bi$. It is interesting to

* We note, for instance, the introduction of the terms *automorphic functions*, *fundamental region*, and the proof and applications of the law of symmetry.

notice that here as in Weber's Algebra the complex number is regarded as a "couple" of real numbers. Thus two of the most recent and most excellent German text-books have returned to the point of view familiar in England more than fifty years ago.

Chapter II on rational functions is of an elementary algebraic and geometric character. The linear function is studied in some detail here, the elliptic, hyperbolic, and parabolic cases being, for instance, treated, although unfortunately no names are given to them.*

Chapter III is devoted to a statement of some definitions and theorems (no attempt being made to prove these latter) concerning real variables and their functions. We fear that most young readers will find many parts of this chapter so difficult as to be of relatively slight service to them. For other reasons also, to which we shall come back later, this chapter seems less happily conceived than the rest of the book.†

Chapter IV deals with single valued analytic functions of a complex variable. Besides the familiar matters on integration and development in power series found in all text-books, will be found Weierstrass's theorem that a uniformly convergent series of analytic functions is analytic and can be differentiated term by term. The admirable proof here given by means of Cauchy's integral formula is I believe

* §12 which is contained in this chapter is destined, I am afraid, to do very great harm to young readers and this in spite of the fact that there is nothing incorrect in it. This section is devoted to the definition and discussion of the symbol $\infty$. This symbol is defined as a new number by the relation $a/o = \infty$. This is of course perfectly allowable, but it is in the highest degree undesirable because in the first place "$\infty$" as thus defined is of no use, nor can it even be regarded as fulfilling the purpose for which it was introduced, viz., to make the operations of addition, subtraction, multiplication, and division always possible. Before "$\infty$" was introduced there was one exception, division by zero was impossible; afterwards there are five exceptions, the operations $0/0$, $\infty/\infty$, $0 \cdot \infty$, $\infty \pm \infty$ being impossible. What is the gain even from the most abstract point of view? In the second place the $\infty$ of analysis is a very different thing from this useless "$\infty$" and there is the greatest danger of confusing the two. The $\infty$ of analysis is a *variable* which increases indefinitely, not a new kind of *constant*. The student is only too prone to think of it as a very large constant and this section will do much to confirm him in this wrong idea.

† A detailed discussion of the contents of this chapter would carry us too far. Two oversights of a somewhat serious nature must however be mentioned. In §25 theorem VI is not correctly stated (Cf. for instance Tannery et Molk: Fonctions elliptiques, vol. 1, §16, p. 23), and in §26 theorem XV is not correct after the definition of a continuous curve contained in theorem X (Cf. the paper by Peano, *Math. Annalen*, vol. 36, p. 157, or for a simpler presentation the paper by Hilbert, *Math. Annalen*, vol. 38, p. 459).

given in only one other text-book.* The chapter closes with a simple form of Mittag-Leffler's theorem and applications to simply periodic functions.

Chapter V is on multiple valued analytic functions. The angle of a complex quantity is first considered as an infinitely multiple valued function and its Riemann's surface is constructed. We believe the author to be quite right in regarding this as the best introduction to the subject of Riemann's surfaces. The function $\log z$ defined as a definite integral is then discussed, and then come the functions $\sqrt{z}$, $\sqrt[n]{z}$ and a few other simple irrational functions.† The chapter closes with a special case of Weierstrass's theorem concerning the expansion of a function in an infinite product.

Chapter VI, entitled "The general theory of functions," introduces the ideas of analytic extension and natural boundaries and the law of symmetry with applications to the conformal representation of rectilinear and some circular triangles on a half plane.

An alphabetical index closes the book.

It will be seen that the author barely touches on the subject of algebraic functions and their integrals which usually play such an important part in books on the theory of functions. Herein we think he has been wise, for this subject is after all a very special one. Apart from this, however, it will be found that in spite of its small size the volume before us touches on more sides of the subject than most other treatises on the theory of functions.‡ From the

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* Demartres, Cours d'Analyse, vol. 2, p. 74.

† On p. 167 the author says: "An algebraic function is not necessarily in itself simpler than a transcendental function," and hereby states an important fact only too often forgotten. I am however unable to agree with the author that the introduction he gives to the function $\sqrt{z}$ by using $\log z$ as an auxiliary function is a desirable one, any more than I should think a teacher of mechanical drawing right who taught, as the first method of drawing a straight line, the use of some linkage, on the ground that the manufacture of a straight ruler involves still more complicated principles

‡ There are, however, some unfortunate omissions. The side of the subject most closely connected with mathematical physics is barely touched upon (pp. 90 and 95); the fundamental theorem that it is always possible to find a function which in a given region is harmonic and has arbitrarily given values on the boundary not being even mentioned. Riemann's theorem that every simply connected region can be represented conformally on a circle ought surely to have been at least stated Moreover, the concrete examples of conformal representation are all too simple to bring out some of the important principles of the subject. The direct discussion of the integral at the end of §72, for instance, would have largely supplied this omission (Cf. Picard, Traité d'Analyse, vol. 2,

point of view of rigor also the volume has great merits. In this respect, however, it seems to the reviewer that a mistake is made here as in almost all other text books on this subject. It is almost universally considered that in a work on the theory of functions absolute rigor should be demanded even though the reader may have been accustomed up to this point to the utmost laxity. This I believe to be fundamentally wrong. The teacher (or the writer of text books) should insist at every stage of the pupil's development on as great rigor as the pupil is then able to appreciate. If this plan were followed elementary text books on the calculus would not abound with statements contrary to common sense (such for instance as that the differential of a function is the limit of its increment), nor, on the other hand, would writers on the theory of functions try to attain absolute rigor at all points. In other words the critical sense of the student should be gradually developed and even if this is properly done it cannot be expected that it will be fully developed when he begins the study of the theory of functions.

If what has just been said is correct, it follows that the attempt made in Chapter III to secure absolute rigor is unwise. Apart from this however it seems unfortunate to concentrate in this very brief form almost all the work contained in the book on the subject of exact analysis, as the reader can hardly fail to get the erroneous impression that when once the results of this chapter have been accepted it is not necessary to be on the lookout for any further difficulties of the same nature, especially as the author has usually arranged his work very skilfully so as to avoid such difficulties. Occasionally however this cannot be done and once or twice the author has overlooked essential points as in Theorem VIII, p. 92, which says that if in a region $S$ of the $z$-plane $w = f(z)$ is a single valued analytic function which does not take on the same value twice and whose derivative does not vanish in $S$, then the values of $w$ fill a region (Bereich) within which $z$ is an analytic function of $w$. Now it is obvious that the region $S$ corresponds in a one to one manner to a set of points (Punktmenge) in the $w$-plane, but how do we know that this set of points is a "Bereich" in the technical sense in which the word is here used? How do we even know that it is a two dimensional continuum, for that is the essential point? I know of only two methods

chap. X., §§ 9, 11). Another subject which might well have been included is the determination of functions satisfying simple functional relations.

which the author would have regarded as available by which this problem can be attacked: *first*, the method suggested by Briot and Bouquet (théorie des fonctions elliptiques §130), or the modification of this method suggested by C. Neumann (Abelsche Integrale, Chap. VI., §2); and *second* the theory of implicit functions of two real variables involving the use of Jacobians.* It is to be regretted that the author did not include a brief and elementary account of this last mentioned method, which has so many other important applications, rather than some of the more difficult parts of Chapter III.

After all has been said however the volume before us remains an excellent treatment of the subject; good as an introduction, in so far as it does not prove too difficult; excellent for the mature student who already knows something of the subject; and invaluable to the teacher.

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## DARBOUX'S ORTHOGONAL SYSTEMS.

*Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes.* Par GASTON DARBOUX, Membre de l'Institut, Doyen de la Faculté des Sciences et Professeur de Géométrie Supérieure à l'Université de Paris. Tome I, Paris, Gauthier-Villars et fils, 1898. 8vo, i+338 pp.

THE present volume is the fifth which Professor Darboux has prepared during the last decade for the Course in Geometry of the Faculty of Sciences of the University of Paris. This new work is to be devoted to the exposition of a theory which has its origin in the writings of Lamé and which has been the subject of a large number of researches in recent years. It is a direct sequel to the author's admirable treatise on the theory of surfaces in which he presented incidentally various properties of orthogonal systems and curvilinear coördinates, but reserved the organic and systematic development of these theories for a separate treatise of which the above is the first volume. The work glistens with originality both in material and in modes of presentation; the exposition exhibits the elegance and clearness characteristic of the author's writings, and the volume, as

* Cf. Jordan, Cours d'Analyse, vol. 1, pp. 80-89.