discussion of the Cartesian oval and its accompanying parabola.

Professor Hall's paper (No. 13) was a continuation of the subject treated in his paper read before the Society at the Boston summer meeting.

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EVANSTON, ILLINOIS.

REPORT ON RECENT PROGRESS IN THE THEORY OF THE GROUPS OF A FINITE ORDER.*

BY DR. G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Boston, August 25, 1898.)

DURING the last decade the general theory of groups has been made much more accessible by means of the publication of a number of treatises. Among these the six volumes by Lie, assisted by Engel and Scheffers (1888–1896), stand out preëminently. The other important treatises that were published in this period are: Cole's translation of a revised edition of Netto's "Theory of Substitutions" (1892); Kantor, "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene" (1895);† Vogt, "Leçons sur la résolution algébrique des équations" (1895); Weber, "Lehrbuch der Algebra" (1895–1897); Burnside, "Theory of Groups of a Finite Order" (1897); Klein-Fricke, "Automorphe Functionen (Die gruppen-theoretischen Grundlagen)" (1897); Bianchi, "Teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Galois" (1897).

A number of other books published during this period

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* The paper was prepared on the invitation of the officers and committee of Section A, "with a view to obtaining at this anniversary meeting such a survey of the field as may lead to a possible coöperation of effort."

† Cf. Wiman, *Math. Annalen*, vol. 48 (1896), pp. 195–240. This article has for its object the complete enumeration of the finite groups of birational transformations in the plane. The results do not agree with those at which Kantor arrived in the treatise cited.

develop this theory less extensively with a view to some direct applications. Among these are: Borel-Drach, "Théorie des nombres et de l'algèbre supérieure" (1895); Picard, "Traité d'analyse, vol. 3" (1896); Page, "Ordinary differential equations; An elementary text-book, with an introduction to Lie's theory of the group of one parameter" (1897). In Picard's Traité the theory of substitutions is introduced with a view towards studying the analogies between the theory of linear differential equations and the Galois theory of algebraic equations. This is a departure from the methods followed in the older works on this subject.

In view of the large number of these recent treatises and the extensive reviews which have been devoted to them in journals that are easily accessible, it seems permissible to confine this report to a few subjects which seem to offer inviting fields for investigation. It is the object to avoid, as far as practicable, the consideration of the recent developments which are fully treated in any of the given works and to aim to supplement a few of them rather than to state what they contain.

We shall generally confine ourselves to the consideration of developments which have been published during the last five years, stating only so much of the earlier results as may seem necessary to give a clear view of the problems. The vastness of our subject may be inferred from the following statement of Lie: *The concepts, group and invariant, take each day a more preponderant place in mathematics and tend to dominate this entire science.** Unexpected demands on the writer's time have compelled him to confine his attention to the subjects which relate to groups of a finite order.

§1. *Solvable groups.*†

A solvable group of composite order must be compound, but a compound group is not necessarily solvable. Sometimes the proof that all the groups of composite orders that belong to a given system are compound is equivalent to the proof that all these groups are solvable. This is clearly the case when the self-conjugate subgroups and the corresponding quotient groups belong to the same system, *e. g.*, when Frobenius proved that every group of a composite order that is not divisible by the square of a prime number must be

* "Influence de Galois sur le développement des mathématiques," *Le Centenaire de l'École Normale*, 1895, p. 485.

† In Weber's recent Algebra these groups are called "metacyclic," vol. 1, p. 598.

compound he also proved that all these groups are solvable, for the orders of their self-conjugate subgroups and of their quotient groups cannot be divisible by the square of a prime number. Similarly, the proof that every group whose order is a power of a prime number is compound proves at the same time the solvability of all these groups.

On the contrary, a group whose order is divisible by 2 without being divisible by 4 must contain negative substitutions when it is represented as a regular group and hence it must be compound when its order exceeds 2. This, however, does not prove that such a group must be solvable, unless we could assume that every group of an odd order is solvable. Burnside has recently called attention to the fact that it would be very desirable to prove either the existence or the non-existence of a simple group of an odd order. If such a group exists it cannot be represented as a doubly transitive substitution group.

From the preceding paragraph we see that the proof that every group of an odd order is compound (if this proof is possible) would prove, at the same time, the solvability of every group whose order is not divisible by 4. It has been observed that the groups of order $p^{\alpha}q^{\beta}$ (p and q being prime numbers) furnish an almost equally interesting field for study. Frobenius proved* that such groups are solvable whenever $\beta = 1$ and Jordan published an article† a few months ago in which he proved that they are also solvable whenever $\beta = 2$. When $\beta = 0$ we have groups whose order is a power of a prime number. Sylow proved‡ a long time ago that all these groups are solvable. Many special cases of groups whose order is of the form $p^{\alpha}q^{\beta}$ have been studied.§

Within the last two years two special systems of solvable groups have been investigated. One of these systems was first considered by Dedekind and has been called by him the Hamiltonian groups‖. It includes all those groups which are non-Abelian and contain only self-conjugate subgroups. The group of lowest order that belongs to this system is the quaternion group, *i. e.*, the group of order 8 which contains 6 operators of order 4. Dedekind found that all the groups of this system have a surprisingly simple structure, viz., each of them is the direct product of the qua-

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* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 185.

† Jordan, *Liouville's Journal*, vol. 4 (1898), p. 21.

‡ Sylow, *Math. Annalen*, vol. 5 (1872), p. 588.

§ Cf. Burnside, Theory of Groups, 1897, Ch. XV.

‖ Dedekind, *Math. Annalen*, vol. 48 (1897), p. 548.

ternion group and an Abelian group which includes only the operators that are commutative to every operator of the group. The writer has studied this infinite system somewhat farther and arrived at a number of new results.* One of the most important of these is that there is one and only one Hamiltonian group of order 2^a, whenever $a > 2$.

A somewhat more extensive system which includes the preceding was studied by Ahrens with a view to pointing out analogies between Lie's theories and the theories of the groups of a finite order.† This system includes all the groups that possess the property that we arrive at identity by forming the successive groups of cogredient isomorphisms. That all the groups of this infinite system are solvable follows directly from the fact that the factors of composition of any group may be found by combining the factors of composition of its group of cogredient isomorphisms and the prime factors of the quotient obtained by dividing its order by that of this group of cogredient isomorphisms.

Although this system includes an infinite number of solvable groups yet there are evidently many infinite systems of solvable groups which do not belong to it. As such systems we may mention the almost trivial systems obtained by forming the direct products of the symmetric groups of degrees three or four and all the possible Abelian groups. From the fact that every group whose order is a power of a prime number includes operators that are commutative to every operator of the group it follows that all these groups belong to the given system.

Since a solvable group that does not belong to this system must either coincide with its group of cogredient isomorphisms, or one of its successive groups of cogredient isomorphisms must have this property, it seems desirable that a special study should be made of the solvable groups which coincide with their groups of cogredient isomorphisms. A large number of these groups may be represented as substitution groups of a low degree, for every solvable group which contains no operator besides identity that is commutative to each one of its operators is clearly a group of this kind.

During the last three years there has been developed a theoretically elegant method to prove the solvability of any given group. It also had its origin in Lie's groups and its

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* Miller, BULLETIN, vol. 4, pp. 510–515.

† Ahrens, *Leipziger Berichte*, vol. 49 (1897), pp. 616–626.

first developments were due to suggestions received from this great investigator. We proceed to give an outline of the method and to indicate some of the related unsolved problems.

If we multiply the inverse of every operator of a group G into all its transforms with respect to the operators of G we obtain a series of operators which form a characteristic* subgroup of G and this is the smallest self-conjugate subgroup of G with respect to which it is isomorphic to an Abelian group. This characteristic subgroup has been called the first derivative or the commutator subgroup of G. The first derivative of the first derivative is called the second derivative of G, etc.

If we arrive at identity by finding the successive derivatives of G it must be a solvable group.† *This condition is necessary as well as sufficient.* That it is necessary follows directly from the facts that a group which coincides with its first derivative cannot be isomorphic to any Abelian group and that every quotient group of a solvable group is solvable. That it is also sufficient is a consequence of the well known theorem that the factors of composition of G may be obtained by combining those of its first derivative and the prime factors of the order of the corresponding quotient group.

It is evident that the subgroup which corresponds to identity of the group of cogredient isomorphisms is an Abelian characteristic subgroup, and that that which corresponds to identity in any one of the successive groups of cogredient isomorphisms is characteristic but not always Abelian. Hence the series of the successive groups of cogredient isomorphisms and that of the successive derivatives have this marked difference, that in one series the quotient group is always Abelian while the corresponding characteristic subgroup does not necessarily have this property, and in the other series the reverse is true. Since this system includes all solvable groups it must include the preceding system.

A group which coincides with its first derivative is called a perfect group. From the above it follows that an unsolvable group is either perfect or one of its derivatives is perfect. Hence it is seen that the study of perfect groups is fundamental in the study of unsolvable groups. It is clear that every simple group of composite order is perfect and that the direct product of any number of such groups

*Frobenius, *Berliner Sitzungsberichte*, 1895, p. 183.
†Miller, *Quar. Jour. of Math.*, vol. 28 (1896), p. 266.

has the same property. The system of perfect groups, therefore, is larger than that of simple groups. Each of these two systems contains the same number of groups that can be represented as transitive substitution groups of a prime degree,* for a perfect transitive group of a prime degree is simple.

The well-known articles of Frobenius "Ueber auflösbare Gruppen,"† seem to the writer to be by far the most important direct contribution to the theory of solvable groups that has been made during the last five years. The theorem that every group whose order is not divisible by the square of a prime number is solvable is especially valuable and it is one of the most general known theorems in this line. The articles contain a number of other useful theorems. Most of these are found in Chapter XV. of Burnside's Theory of Groups. This excellent chapter contains also some new theorems by the author of the book.

With respect to the groups of a low order (and probably also with respect to those of a high order) solvability is the rule and unsolvability is the exception. This also applies to the groups which may be represented as substitution groups of a small degree. Hölder has recently enumerated all the unsolvable groups whose order does not exceed 479. The following table contains the orders of these twenty-five groups :‡

Order	60	120	168	180	240	300	336	360	420
Number	1	3	1	1	8	1	3	6	1

It is well known that the degree of a primitive solvable group is a power of a prime number. The following table§ gives the numbers of all these groups whose degree is less than 25 :

Degree	3	4	5	7	8	9	11	13	16	17	19	23
Number	2	2	3	4	2	7	4	6	10	5	6	4

The following table gives the numbers of the insolvable groups which may be represented as substitution groups whose degree is less than 12 :||

Degree	5	6	7	8	9	10	11
Number	2	4	6	16	41	106	228

* Miller, *Amer. Jour. of Math.*, vol. 20 (1898), p. 277.

† Frobenius, *Berliner Sitzungsberichte*, 1893–95, pp. 337, 1027.

‡ Hölder, *Math. Annalen*, vol. 46 (1895), p. 420.

§ Miller, *Amer. Jour. of Math.*, vol. 20 (1898), p. 241.

|| Ibid., p. 282.

§ 2. *Simple groups.*

We have already observed that there is an intimate relation between the questions of solvability and simplicity. In fact, if there were no simple groups of a composite order every group would be solvable. The simple groups also play a fundamental rôle in other domains of the theory of groups and it is a matter of the greatest interest that Killing and Cartan have recently succeeded in proving that there is no finite continuous simple group besides the five groups which have respectively 14, 52, 78, 133 and 248 parameters and the four great systems established by Lie.*

In the theory of discontinuous groups (finite and infinite) as well as in that of the infinite continuous groups the enumeration of all the possible simple groups seems to offer much greater difficulties. Although infinite systems of such groups are known, yet these do not give any assurance that a complete enumeration can ever be made, since the general theory of these groups is very complex. In what follows we shall confine our attention to the simple groups of a finite order.

The division of groups into simple and compound is due to Galois who also observed that there is no simple group of any composite order less than 60.† About six years ago Hölder proved that there are only two simple groups of a composite order less than 200, viz., the well-known groups of orders 60 and 168.‡ Soon after this Cole considered all the possible simple groups whose orders are composite and lie between 200 and 661. He found that there are only three such groups, viz., one for each of the orders 360, 504 and 660.§

The simplicity of the group of order 504 was discovered by Cole. This is the lowest order for which a simple group has been discovered within the last decade. The historical interest of this group has been greatly enhanced by the fact that it served as a starting point for some very interesting investigations by Moore which led to the proof of the existence of a doubly infinite system of simple groups. The given group may be represented as a transitive substitution group of degree 9 since it contains a subgroup of order 56. Kirkman gave it in this form in his list of the transitive groups whose degree does not exceed ten,‖ but he appar-

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* Cf. Cartan, Thesis, Paris, 1894.

† Lettre de Galois à M. Chevalier, *Liouville's Journal*, vol. 11 (1846), p. 409.

‡ Hölder, *Math. Annalen*, vol. 40 (1892), pp. 55–88.

§ Cole, *Amer. Jour. of Math.*, vol. 15 (1893), pp. 303–315.

‖ Kirkman, *Proc. Lit. and Phil. Soc. of Manchester*, vol. 3 (1863), pp. 133–152.

ently did not know that it is simple.

Burnside has continued the search for simple groups of a low order by investigating the orders from 661 to 1092. He found only one simple group whose order lies in this region, viz., the well-known group of order 1092 which may be represented as a transitive group of degree 14. This belongs to the system of groups of the modular equation for the transformation of elliptic functions. If we remember that Cole found that the given group of order 504 is simple while seeking all the possible groups that can be represented as transitive substitution groups of degree 9*, we see that only negative results have been reached by the study of all the possible simple groups whose order is less than 1093. Regardless of this fact it seems desirable that the search of simple groups along this line should be continued somewhat further.

During the last few years very interesting systems of simple groups have been obtained by means of the application of an extension of the elementary concept of congruences, or the theory of the Galois field whose order is a power of a prime number. Moore proved a fundamental theorem in regard to the theory of this field, viz. : *Every existent field is the abstract form of a Galois field whose order is a power of a prime number*;† *i. e.*, every existent finite field of quantities such that the sum, difference, product and quotient (except by 0) shall belong again to the field must be a Galois field of order a power of a prime.

By the application of this theory Moore arrived at the doubly infinite system of simple groups which was mentioned above, and which is contained in the same article as the given theorem. A few months later Burnside was led to the same system by means of somewhat similar considerations.‡ Still further generalizations have led Dickson and Burnside to triply infinite systems of simple groups. The first of these systems was found by Dickson while working on his Chicago thesis.§ Since this time Dickson has arrived at other systems of simple groups|| and he is continuing his important investigations in this direction.

It is an interesting fact that nearly all the known simple groups belong to one or more of the known infinite systems.

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*Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1892), pp. 253–4.

†Moore, Chicago Mathematical Papers, 1893, p. 211.

‡Burnside, *Proc. Lond. Math. Soc.*, vol. 25 (1894), pp. 113–139.

§Dickson, *Annals of Mathematics*, vol. 11 (1897).

||Dickson, *Quar. Jour. of Math.*, vol. 29, pp. 169–178.

The groups of orders 7920 and 95040 whose minimum degrees when they are represented as transitive substitution groups are 11 and 12, respectively, seem to be exceptions.* This, however, is no index that they do not form a part of some such systems that have not yet been developed. The fact that there are isolated cases in the finite continuous groups might perhaps lead one to suspect that similar cases might occur among the groups of a finite order.

It is evident that we can always construct more than one group of a given composite order except when none of its prime factors diminished by 1 or 0 is divisible by another one of them. In this case (as well as in the case when the order is a prime number) we can construct only the cyclical group. As it is not possible to construct more than one simple group for each of the lowest orders for which such groups are possible, it was a question of considerable interest to inquire whether it is possible to construct two simple groups of the same order and what is the lowest order for which this is possible. The first part of this question has just been answered in the affirmative by Miss Schottenfels of Chicago University. She succeeded in proving that there is at least one simple group of order $8! \div 2$ which is not simply isomorphic to the alternating group of degree 8.

A search for simple groups has also been made among the groups whose orders involve a small number of prime factors. It is evident that every group whose order is a prime number is simple. In seeking all the groups whose orders are composite and contain no more than three prime factors Netto, Cole and Glover, and Hölder observed that none of these groups are simple. Soon after this Frobenius observed that the group of order 60 is the only simple group whose order is composite and contains no more than four prime factors. This is clearly equivalent to the proof that all the other groups whose orders contain no more than four prime factors are solvable, for an unsolvable group must either be isomorphic to a simple group of composite order (which may be the group itself) or it must contain a characteristic subgroup that has this property.

Burnside has carried investigations along this line somewhat farther in his "Notes" published in the *Proceedings of the London Mathematical Society*, 1894–5. He proved, among other things, the correctness of a remark by Frobenius to the effect that there are only three simple groups whose

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*BULLETIN, vol. 4 (1898), pp. 382–389.

order is the product of five prime numbers, viz., one group of each of the orders 168, 660, and 1092. The possible simple groups whose orders are products of six or a larger number of prime factors have not yet been determined. While the difficulty of this problem increases rapidly with the number of the prime factors, yet it seems quite desirable that investigations along this line should be continued.

We observed above that the study of congruences has led to remarkable systems of simple groups. This study has been by far the most helpful in determining such groups. The study of substitutions has led to a number of individual simple groups as well as to the well known system of alternating groups whose degree exceeds 4. Within the last year Beke has published a new proof of the simplicity of the groups of this system.* This proof is based upon very elementary concepts. Since every self-conjugate subgroup of a transitive group of a prime degree must include all its substitutions of a prime order, these substitutions must always generate a simple group, which is evidently of a composite order whenever the group contains more than one subgroup of order p. Hence every transitive group of a prime degree that is not included in the metacyclic group of this degree contains a simple group of composite order as self-conjugate subgroup. This may be the group itself. The quotient group with respect to this simple group is always cyclical, *i. e.*, such a group contains only one composite factor of composition.

While the proof of the existence of simple groups is of great interest, yet it leaves many interesting questions in regard to these groups unsolved. One of the most important of these is the determination of their subgroups. In fact, it might be said that the entire theory of the groups of a finite order is involved in the subgroups of the system of simple groups which is composed of the alternating groups; for every group whose order does not exceed n is clearly a subgroup of the alternating group of degree $n + 2$.

In a paper read at the last summer meeting of the AMERICAN MATHEMATICAL SOCIETY, Moore gave a determination of all the subgroups of the generalized modular group of a finite order. This is an extension of the noted paper by Gierster† in which he determined all the subgroups of the modular group and found that it can contain only three kinds of subgroups besides the cyclic and the metacyclic ones. The generalized modular group was first studied by Mathieu

* Beke, *Math. Annalen*, vol. 49 (1897), p. 581.

†Gierster, *Math. Annalen*, vol. 18 (1881), pp. 319-365.

with regard to its cyclical subgroups. In an earlier paper Moore has proved that it is simple except for a few special cases,* and in the present paper he gives a complete determination of its subgroups. He has kindly favored me with the following important statements in regard to this group:

The modular group Γ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

of the complex variable ω, where α, β, γ, δ are rational integers, has for every rational prime q a self-conjugate subgroup $\Gamma_{\mu(q)}$ of finite index $\mu(q)$ containing all substitutions $(\alpha, \beta, \gamma, \delta)$ for which $\alpha \equiv 1$, $\beta \equiv 0$, $\gamma \equiv 0$, $\delta \equiv 1$ (mod q). The corresponding quotient group is conveniently given as the finite modular group $G^{q+1}_{\mu(q)}$ of substitutions $(\alpha, \beta, \gamma, \delta)$ on the $q + 1$ marks, ∞, $0, 1, \cdots, q - 1$, where the α, β, γ, δ are integers taken modulo q. By generalizing from the Galois field of rank 1 to that of rank n we have the generalized finite modular group $G^{q^n+1}_{M(q^n)}$ of order $M(q^n) = q^n (q^{2n} - 1)$ or $q^n (q^{2n} - 1)/2$. Its subgroups are of three kinds: (1) metacylic or solvable groups, (2) and (3) groups of the abstract character of certain groups $G^{q^n+1}_{M(q^n)}$, or of certain groups $G^{q^n+1}_{2M(q^n)}$, $q > 2$; the latter group being obtained by extending the former by the substitution $\omega' = \rho\omega$, where ρ is a primitive root of the Galois field of order q^n.

Thus the doubly infinite system of simple groups $G^{q^n+1}_{M(q}$ $(q^n \neq 2^1, 3^1)$ determines by the decomposition of the subgroups of its constituent groups, apart from the simple groups of prime order, only simple groups of the original system. An equation of degree $q^n + 1$ $(q^n \neq 2^1, 3^1)$ whose Galois group is the $G^{q^n+1}_{M(q^n)}$ has resolvents of degree D, $D < q^n + 1$, only in the cases $q^n = 5^1, 7^1, 11^1, 3^2$, when D is respectively 5, 7, 11, 6. For the case $n = 1$, this is a noted theorem of Galois and Gierster.

The group $\Gamma[\Omega]$ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$ where the coefficients are algebraic integers of an algebraic field Ω [Körper Ω] has, for every prime functional Π [Weber] of absolute norm q^n, a self-conjugate subgroup $\Gamma_{M(q^n)}[\Omega]$. The quotient group is the group $G^{q^n+1}_{Mq^{(n)}}$.

*Moore, Chicago Mathematical Congress Papers, 1893, pp. 208–242. This is the system of simple groups which was mentioned above, p. 233.

The substitutions of the modular group Γ leave invariant a modular function $J(\omega)$. The transcendental modular equation $J = J(\omega)$ has algebraic resolvents, modular equations, with Galois groups $G^{q+1}_{M(q)=\mu(q)}$. The group $\Gamma[\Omega]$ for a field Ω of degree > 1 is improperly discontinuous and so has no corresponding automorphic function. But if an automorphic function $K(\omega)$ belongs to a group whose substitutions belong to an algebraic field Ω, then the generalized modular equation $K = K(\omega)$ has algebraic resolvents whose Galois groups are certain generalized finite modular groups $G^{q^n+1}_{M(q^n)}$ or certain of their subgroups. This remark connects the present paper with recent work of Fricke and Bianchi on certain special classes of automorphic functions (*cf.* Klein-Fricke, Automorphe Functionen, vol. 1, 1897, p. 586 fg.).

§ 3. *Substitution groups.*

In what might probably be called the earliest work on groups, viz., "Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebrica delle equazioni generali di grado superiore al quarto," di Paolo Ruffini, Bologna, 1799, the substitution groups are divided into transitive and intransitive as well as into primitive and imprimitive groups. These divisions have been adopted by all the subsequent writers. The primitive groups present the greatest difficulties and they have therefore received most attention.

One of the most interesting problems in regard to the primitive groups which do not include the alternating group is the determination of the smallest degree of any of its substitutions that differ from identity. Jordan has called this minimum degree the class of the primitive group and he was the first to make important investigations along this line.* In fact, Jordan's work and influence in this direction have been so preëminent that it would seem proper to call the given problem Jordan's problem.

During the last few years Bochert, Jordan, and Maillet have made important progress towards the solution of this problem. The work of Bochert is contained in a series of articles published in the *Mathematische Annalen*.† These articles are based upon very elementary principles. One of the most important of these is a direct consequence of the commutator of two substitutions and it may be stated as

* Cf. Jordan, *Liouville's Journal*, vol. 16 (1871).

† Vols. 29, 33, 40, 49.

follows: *If a group contains two substitutions that are not commutative it must also contain a positive substitution whose degree does not exceed three times the number of common elements of the two given substitutions.*

A special case of this principle, which can frequently be employed, is the fact that the commutator of two substitutions which have only one common element is always a substitution of degree 3. While Jordan, Sylow, and Bochert employed the commutator of two substitutions as an instrument, yet none of them seems to have made a special study of the properties of this operator. We have seen that this operator has some applications in substitution groups besides those which relate directly to the abstract group properties and hence it seems very desirable that its properties should be more completely determined. For instance, it would be very desirable to find a method by means of which the commutator of two substitutions could be obtained almost as readily as their transforms with respect to each other,* or a method by means of which we can easily see the form of the commutator whenever the substitutions are given. It would also be interesting to find out whether the commutators of the operators of a group constitute all the operators of the subgroup which they generate; *i. e.*, whether all the operators of the first derivative are commutators of operators in the group.

While the commutator of two substitutions is more complex than the transform of one of them with respect to the other, yet it sometimes has a decided advantage on account of its more nearly symmetric form, the interchange of the two substitutions giving merely the inverse of the commutator. If we permute the four operators or substitutions of any commutator in every possible manner we obtain eight commutators which may be distinct operators;† each of the other 16 forms is identically equal to unity, hence none of these 16 forms is a commutator. The transitive group of degree 4 and of order 8 is therefore the largest group of this degree that transforms a commutator into a commutator by merely permuting its operators.

The most practical theorem that Bochert found in the given series of articles may be stated as follows: *If a group of degree n does not include the alternating group and is more than simply transitive its class exceeds $\frac{1}{4}n - 1$, if it is more than doubly transitive its class exceeds $\frac{1}{3}n - 1$, and if it is more than*

* Jordan, Traité des substitutions, 1870, p. 23.

† Cf. Bochert, *Math. Annalen*, vol. 40 (1892), p. 159.

triply transitive its class is not less than $\frac{1}{2}n - 1$.* Jordan regarded this theorem as the most important advance made since the theorem of Sylow was published.† It should, however, be observed that this theorem differs widely from that of Sylow in regard to applications; while Sylow's theorem applies to groups regardless of the notation, that of Bochert is directly applicable only to primitive substitution groups. In a recent article published in *Liouville* ‡ Jordan succeeded in obtaining somewhat closer limits for the class than those that had been obtained by Bochert.

Maillet has published a large number of articles in which the class of a primitive group is considered. His memoir§ entitled, "Recherches sur la classe et l'ordre des groupes de substitutions" was crowned in 1896 by the Paris Academy with the Grand prix des sciences mathématiques. In the first part of this memoir the author treats the class of primitive groups which are isomorphic to the symmetric or the alternating group. In the second part he considers the highest order of a primitive group that does not include the alternating group. Both of these questions are old and they have been the subjects of a large number of memoirs. Maillet attacked them from new standpoints and succeeded in finding new and interesting results.||

It is well known that the metacyclic group of degree p, p being any prime number, is of class $p - 1$. The question naturally arises in what other primitive groups is the class equal to the degree diminished by unity. Maillet has devoted a great deal of attention to this question and he succeeded in proving that when the degree of a primitive group is less than 202 its class cannot be obtained by diminishing the degree by unity unless the degree is a power of a prime number.¶ In his thesis he also considered the primitive groups of degree n and classes $n - 2$ and $n - 3$**.

The question of the class of a primitive group is very closely related to that of its limit of transitivity and these two questions have frequently been studied together. In Vol. I of the *Bulletin de la Société Mathématique de France* Jordan published several elegant theorems on the limit of transitivity. It is a singular fact that these theorems have not found a place in the text books, especially the one which

* Bochert, *Math. Annalen*, vol. 40 (1892), p. 179.
† Jordan, *Liouville's Journal*, fifth series, vol. 1 (1895), p. 35.
‡ *Ibid.*
§ *Mém. sav. étr.*, vol. 32, no. 8.
|| *Cf.* Jordan, *Comptes rendus*, vol. 123 (1896), pp. 1107–1109.
¶ Maillet, *Bull. de la Soc. Math. de France*, vol. 25 (1897), pp. 16–32.
** Maillet, Thesis, *Paris*, 1894.

asserts that *a group of degree $p+k$, p being any prime number, cannot be more than k times transitive whenever $k>2$.* Quite recently the writer has given a somewhat different proof to two of these theorems and he has extended one of them. This proof is based upon the lemma (which does not appear to have been noticed) that *the quotient group of any transitive group of a prime degree with respect to any self-conjugate subgroup (except identity) is cyclical.**

The determination of all the groups of degree n has always been a prominent goal in the theory of substitutions and it has been called the principal object of the theory of substitutions.† Although this problem has been attacked by a large number of mathematicians yet it still remains far from a complete solution. For small degrees the problem can be solved very easily but the difficulty increases rapidly with the degree. We proceed to give a short account of recent progress in this direction.

In Vol. 24 of the *Quarterly Journal of Mathematics*, Askwith published two articles in which he investigated the groups whose degree does not exceed eight. In the following volume of this journal Cayley gave the results of Askwith together with some additions. Numerous omissions occur in this list, even with respect to groups that were published at an earlier date. A supplementary list together with some interesting methods was published by Cole about a year later.‡ This was followed by numerous additional articles § by Cole and the writer in which the intransitive, the imprimitive, and the primitive groups are given up to degrees 11, 15, and 17 respectively. In several of these papers there is no attempt to give anything besides the results.

In these enumerations it frequently happens that the same abstract group is given more than once. This does not cause any inconvenience in regard to the transitive groups, for if we know the properties of all the subgroups of a given group we know definitely in how many ways it may be represented as a transitive substitution group. Hence the determination of all the possible transitive groups of a given order is equivalent to the determination of all the abstract groups of this order and the study of some of the properties of these groups. When a group is simple

* Miller, BULLETIN, vol. 4 (1898), p. 141.

† Serret, Algèbre supérieure, 5th edition, vol. 2, p. 283.

‡ Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1893), p. 184.

§ These articles have appeared in recent numbers of the BULLETIN, *Quar. Jour. of Math.*, *Proc. Lond. Math. Soc.*, and *Amer. Jour. of Math.*

the determination of all its different sets of subgroups such that each set includes all those that may correspond in a simple isomorphism of the group to itself is equivalent to the determination of all the different transitive forms in which it may be represented as a transitive substitution group.

There is no such simple relation between the abstract groups and the ways in which they can be represented as intransitive substitution groups. Very little has been done towards giving formulas by means of which all the possible intransitive substitution groups of a given order and any arbitrary degree may readily be determined. Such formulas have recently been published for the cases when the order is 4 or when it is the product of two unequal prime numbers.* The next step in this direction will probably be a formula by means of which we can directly ascertain the number of the intransitive groups whose order is the square of a prime and whose degree is an arbitrary number. It is evident that the total number of intransitive groups of any given order is infinite, while that of transitive groups is finite whenever the order is finite.

While a large number of the substitution groups have been found by tentative methods yet it is very easy to establish general theorems by means of which all trials may be avoided, at least, when the degree is small. It seems, however, that the number of the necessary general theorems increases rapidly with the degree. The known theorems are sufficient to determine all the possible substitution groups at least as far as degree nine without any trials. The present theory of the imprimitive groups is perhaps more unsatisfactory than that of the other two divisions of substitution groups.

It is frequently desirable to express substitutions analytically. In his Chicago thesis of 1896, Dickson has considered (among other things) the determination of all quantics up to as high a degree as practicable which are suitable to represent substitutions on p^n letters, p being any prime number, as well as the determination of special quantities suitable on p^n letters where for each quantic the combination (p, n) takes infinitely many values. He established a number of interesting theorems and added very materially to our knowledge in regard to this subject.

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* Miller, BULLETIN, vol. 2 (1896), pp. 332–334.

§4. *Abstract groups.*

Cayley called attention to the fact that a group is defined by means of the laws of combination of its symbols.* Such a group is called an abstract or operation group. In other words, all simply isomorphic groups are merely different representations of the same abstract group. Every concrete group (such as a substitution group) represents some abstract group and every abstract group may be represented by many concrete groups. All the abstract groups might, therefore, be studied by means of a system of concrete groups selected in such a manner that one, and only one, belongs to each set of simply isomorphic groups. The regular substitution groups, as is well known, form such a system.

This method of study would have the disadvantage that in concrete groups the generational relations are not always defined with clearness, *i. e.*, the concrete generators sometimes involve conditions which do not stand out prominently in the concrete group. Hence it is frequently desirable to study abstract groups without any reference to any particular representation. This requires much greater care as to definition than is necessary for concrete groups, *e. g.*, any set of substitutions, such that no two of them are identical, obeys all the conditions of a group except the condition that the product of any two of them, or the square of any one, should be contained in the set. Hence the definition of a substitution group does not need to contain any other condition and it is therefore exceedingly simple.

During recent years accurate definitions of an abstract group have been formulated. These are principally due to Frobenius and Weber. Such definitions are stated at the beginning of the second volume of Weber's Algebra. It is to be hoped that practical uniformity in regard to the definition of an abstract group may be attained and that the energies of a large number of investigators may thereby be directed towards the same difficulties. While the term abstract group is frequently defined very inadequately, even in recent works, yet this deficiency is generally supplied by the inherent properties of the concrete operators of the groups which are actually used.†

Although a clear insight into the generational relations of a concrete group does not always occupy a prominent place in the study of these groups, yet it is an important part of

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* Cayley, *Amer. Jour. of Math.*, vol. 1 (1878), p. 51.

† Cf. Lie, *Le Centenaire de l'École Normale*, 1895, p. 485.

their study. Some important group properties follow directly from these relations; *e. g.*, if the generators of a group satisfy all the generational relations of another and can be placed in 1, 1 correspondence with them the first of these groups is a quotient group of the second.* We thus arrive directly not only at self-conjugate subgroups of the given group but also at the corresponding quotient groups.

Hence we observe the importance of finding the generational relations of known systems of concrete groups. This seems to offer a desirable field of investigations. Quite recently these relations were determined for the symmetric and the alternating group.† For the Abelian groups they have been known for a long time and for the Hamiltonian groups they follow directly from the known properties of these groups. When the order of a group is the product of a small number of prime factors the generational relations were determined at the same time as the possible groups of these orders were determined.‡ In fact, the generational relations furnish one of the most satisfactory means of determining all the groups whose orders involve a small number of prime factors, while they do not appear to furnish the most direct means to determine all the groups which may be represented as substitution groups of a low degree.

The groups of order p^α, p being any prime number, are of especial interest since every group whose order is divisible by p^α must contain at least one subgroup of this order. Any progress in regard to the theory of these groups is therefore of more than ordinary importance. Among the most important theorems in regard to these groups are the following: Every group of order p^α is solvable and it contains a self-conjugate subgroup of order p^β, whenever $\beta < \alpha$. A subgroup whose order is less than p^α is self-conjugate in a subgroup which includes operators that are not contained in the former of these subgroups. Every operator of the group is commutative to at least p operators of the group.

The following general theorem was published during last year and it includes some other well known theorems in regard to these groups:§ *A group of order p^α contains a subgroup of order $p^{\alpha-\frac{\beta(\beta-1)}{2}}$, such that each of the operators of this*

* Moore, *Proc. Lond. Math. Soc.*, vol. 28 (1896), p. 359.

†Ibid.

‡ Cf. Hölder, *Math Annalen*, vol. 43 (1893), pp. 335–360.

§Miller, *Messenger of Mathematics*, vol. 27 (1897), p. 120.

subgroup is commutative to every one of the operators of a subgroup of order p^β that is contained in the given subgroup. As an important corollary we have: *A group whose order exceeds $p^{\frac{n(n-1)}{2}}$ contains a commutative subgroup of order p^n.* If we let $n = 2$ we have the theorem that a group whose order is p^2 must be Abelian and if $n = 3$ we have that every group whose order is p^4 contains an Abelian subgroup of order p^3.

In the early development of the theory of groups a large number of the theorems were proved by means of substitution groups. It soon became desirable to give proofs which are independent of any particular notation. One of the first important steps in this direction was taken by Frobenius in his proof of Sylow's theorem.* In a more recent paper the same author has proved the important theorem that every simple isomorphism of an abstract group † to itself can be obtained by transforming the group. This theorem is equally true of regular substitution groups but it is not true in regard to all substitution groups, *e. g.*, the modular groups of degrees 7 and 11 are known to have simple isomorphisms to themselves which cannot be obtained by transforming the substitution groups.

While it is a matter of interest to give proofs which are independent of any special notation yet it does not appear desirable to do this at the expense of clearness, *e. g.*, if we make a regular group simply isomorphic to itself we may suppose that all the substitutions begin with the same element. The second elements of the corresponding substitutions give then the substitution which transforms the given regular group into the given isomorphism to itself. Hence we observe that if a regular group of degree n is transformed into itself by the largest possible group of this degree, the subgroup of this group which contains all its substitutions that do not contain a given element is simply isomorphic to the group of isomorphisms of the given regular group. It seems that some questions in regard to the group of isomorphisms can be studied most conveniently by means of this subgroup. The most elementary concept of the group of isomorphisms is that of a substitution group whose elements correspond to the operators of the group which are not characteristic, *i. e.*, to those operators which do not correspond to themselves in every simple isomorphism of the group to itself. It is evident that identity is a characteristic operator of every group. If a group is Abelian and

* Frobenius, *Crelle*, vol. 100 (1887), p. 179.

† Frobenius, *Berliner Sitzungsberichte*, 1895, p. 184.

contains only operators of order p in addition to unity, its group of isomorphisms regarded as such a substitution group is transitive and there is only one characteristic operator.* In all other cases this group of isomorphisms is either intransitive or there is more than one characteristic operator, *e. g.*, the group of isomorphisms of the quaternion group is transitive but there are two characteristic operators.

While the group of isomorphisms may be regarded as a substitution group yet it seems to be more commonly regarded as the abstract group which is simply isomorphic to the given substitution group. As we are chiefly concerned with the group properties of the group of isomorphisms it may be desirable to emphasize this fact by calling it an abstract group. This also avoids the trouble of selecting any particular notation. If we remember that the abstract group properties should form the most prominent part of a substitution group, and that the particular notation should tend to deepen our insight into the abstract group rather than towards making it more superficial, it may not appear essential to call the group of isomorphisms an abstract group.

It is frequently desirable to know all the possible abstract groups of a given order. We have already observed that all the possible groups of any order that do not involve more than three prime factors are known. Only special cases have been investigated when the order involves more than three prime factors. The most important of these has recently been considered by Hölder, who succeeded in finding all the groups whose orders are not divisible by the square of any prime number.† His investigations are based upon results obtained by Frobenius when he proved that all these groups are solvable.

The other extreme case, viz., when the order of the groups is a power of a single prime number, seems to present much greater difficulties. Young and Hölder found at almost the same time all the groups whose order is the fourth power of a prime. Quite recently Bagnera has considered all the possible groups whose order is the fifth power of a prime number.‡ The difficulties of this problem seem to increase very rapidly with the index of the power of the prime number.

The results of this article by Bagnera do not appear to be entirely correct, at least as far as they relate to the groups of order 32. The number of these groups that are Abelian

* Moore, BULLETIN, vol. 2 (1895), p. 33.

† Hölder, *Göttinger Nachrichten*, 1895, pp. 211–229.

‡ Bagnera, *Annali di Matematica*, 1898, pp. 137–228.

or contain a subgroup of order 8 whose operators are self-conjugate in the entire group is correctly stated, but the numbers of those that contain only 2 or 4 operators that are self-conjugate in the entire group are given incorrectly, the number of the former being only 10 instead of 12, and of the latter 19 instead of 16. As there are evidently 7 Abelian groups of this order and 15 that contain 8 operators which are commutative to every operator of the group, the total number of these groups is 51.

Among the much more special cases that have been considered are all the groups whose order does not exceed a given number. This problem has recently been solved for all the orders less than 64, but the results for the orders which exceed 47 have not yet been published. In regard to the groups of order 32, Bagnera's recent article fixes the number of the possible groups of this order at 50 while the actual number seems to be 51 as given above. Levavasseur had supposed their number to be beyond 75.* When the number of the possible groups is so large the probability of making an error is much greater than when the number is small and it is therefore very desirable that such questions should be investigated by independent methods.

A few other very special cases of the general problem of finding all the groups whose order is the product of four prime numbers have been incidentally considered, viz., the orders $8p$† and $2p^3$. These cases are, however, too special to have much influence on the general problem. The complete solution of this problem would be a matter of considerable interest and it seems to be the next desirable step in this line of work.

Conclusion.

"In recent years many groups of permutations of 6, 7, 8, 9, ··· letters have been made known. The problem would be to determine in each case the minimum number of variables with which isomorphic groups of linear substitutions can be found. Secondly, I want to call your particular attention to the case of the general equation of the eighth degree. I have not been able in this case to find a material simplification, so that it would seem as if the equation of the eighth degree were its own normal problem. It would no doubt be interesting to obtain certainty on this point."‡

The first of these problems seems to be still a standing

* Levavasseur, *Comptes rendus*, vol. 122 (Jan., 1896); cf. Miller, *Comptes rendus*, Feb., 1896.

† Miller, *Phil. Magazine*, vol. 42 (1896), p. 195.

‡ Klein, The Evanston Colloquium, 1894, p. 74.

problem while the second has recently been solved by Wiman, who proved more than the given problem by showing that there is no material simplification for the general equation (the symmetric and the alternating substitution group) whose degree exceeds 7.* It may be remarked in passing that Wiman's footnote on page 58 is quite erroneous, the doubly transitive group of order 56 and degree 8 being found in the much earlier list by Kirkman† while the triply transitive group of order 1344 and degree 8 is found in a number of different places in addition to Kirkman's list.

The linear groups are of extreme importance on account of their numerous direct applications. Every group of a finite order can clearly be represented in many ways as a linear substitution group since the ordinary substitution (permutation) groups are merely very special cases of the linear groups. The general question of representing such a group with the least number of variables seems to be far from a complete solution. It is closely related to that of determining all the linear groups of a finite order that can be represented with a small number of variables.

Klein was the first to determine all the finite binary groups‡ while the ternary ones were considered independently by Jordan§ and Valentiner.|| The latter discovered the important group of order 360 which was omitted by Jordan and has recently been proved simply isomorphic to the alternating group of degree 6.¶ Maschke has considered many quaternary groups and established, in particular, a complete form system of the quaternary group of 51840 linear substitutions.**

Frobenius has recently published an article on the representation of finite groups by linear substitutions, which contains a large number of new results.†† This is a continuation of his article on the prime factors of the group determinant.‡‡ In the latter article the author has associated with every finite group H of order h a matrix of degree h whose elements are dependent upon h variables. In the

* Wiman, *Göttinger Nachrichten*, 1897, pp. 53 and 191.

† Kirkman, *Proc. Lit. and Phil. Soc. of Manchester*, vol. 3 (1863), pp. 133–152.

‡ Klein, *Math. Annalen*, vol. 9 (1875), p. 183.

§ Jordan, *Atti delle Reale Accademia di Napoli*, 1880.

|| Valentiner, "De endelige Transformations-Gruppers Theori," 1889.

¶ Wiman, *Math. Annalen*, vol. 47 (1896), pp. 531–556.

** Maschke, *Math. Annalen*, vol. 33 (1889), pp. 317–344; cf. articles by the same author in vols. 30 and 36.

†† Frobenius, *Berliner Sitzungsberichte* (1897), pp. 994–1015.

‡‡ Ibid., 1896.

present article he points out, among other things, the connection between this matrix and the linear substitutions by means of which H and its isomorphic groups may be represented.

In a recent number of *Liouville* Laurent published an "Exposé d'une théorie nouvelle des substitutions,"* in which he proposes to create an algorithm by means of which the general theory of substitutions may be presented. He employs not only the product and powers of substitutions but also their sums and differences. The term *new theory* should perhaps be understood to mean that the author is dealing mostly with facts that have not appeared in treatises. References would seem to have made the article more useful. The author adds : "Ceux qui voudront bien lire les pages qui suivent se convaincront que je n' ai fait qu' effleurer un sujet très vaste."

In closing we would repeat what was stated at the beginning of this report that we have aimed to call attention to only a few of the important recent advances in the theory of groups. In almost all parts of higher mathematics the group theory is continually taking a more prominent position† and it would require a man of riper years and much wider attainments than those possessed by the writer to give a harmonious and extensive account of the marvelous recent progress in this field. If our humble efforts shall be of service to some beginner in leading him to problems whose solution will assist him to penetrate the rich fields of this theory they will be amply rewarded.

CORNELL UNIVERSITY,
August, 1898.

NOTE ON BURNSIDE'S THEORY OF GROUPS.

BY DR. G. A. MILLER.

IT is well known that Professor Cayley published an enumeration of the possible substitution groups whose degree does not exceed eight‡ and that Professor Cole pub-

* Laurent, *Liouville's Journal*, vol. 4 (1898), pp. 75-119 ; cf. recent articles by the same author in *Nouvelles Annales*.

† Klein-Fricke, Vorlesungen über die Theorie der automorphen Functionen, 1897, p. 1.

‡ Cayley, *Quar. Jour. of Math.*, vol. 25 (1891), pp. 71 and 137.