ASYMPTOTIC LINES ON RULED SURFACES HAVING TWO RECTILINEAR DIRECTRICES.

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LET l be a generator of a non-developable ruled surface S, and let π_s be the tangent plane to S at a point P on l. As P moves along l, π_s will turn about l in such a way that the range (P) and the axial pencil (π_s) are homographic (Chasles's correlation).

Again, let l be a line of a linear complex C, and let π_c be the polar plane in C of a point P on l. As P moves along l, π_c will turn about l in such a way that the range (P) and the axial pencil (π_c) are homographic (normal correlation).

The two axial pencils (π_s) and (π_c), being homographic with a common range (P), are projective with each other when l belongs to S and to C. These two projective pencils have two self-corresponding planes, π_1 and π_2, such that the point of tangency and pole coincide; let the corresponding points be P_1 and P_2.

If all the generators l of S belong to C, there will be two points on each, such that the tangent plane and the polar plane coincide. The locus of these points will be a curve traced on the surface, called the *complex curve*.

Let P' be a point on the curve contiguous to P, then PP' is a line of C, hence all the tangents to the curve belong to the complex C. The complex curve is an asymptotic line on the surface, because all its osculating planes are also tangent planes, a characteristic property of the asymptotic lines. This theorem may be stated as follows: *Every ruled surface contained in a linear complex has an asymptotic line, all of whose tangents belong to the complex.*

When the surface is algebraic, the order of this asymptotic line can be easily determined.* Consider any plane α and let it cut the curve in a point K; the polar plane of K with regard to C, which is also the tangent plane to the surface at K, will pass through A, the pole of the plane α. But every line of this polar plane $\varkappa$ of K is a tangent to the sur-

* Picard, "Mémoire sur une application de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches." *Annales de l'École Normale*, 1877.

face, hence KA is tangent at K to the section of the surface made by α.

Conversely, let K be the point of contact of a tangent to the curve of this section from the pole A. The plane passed through KA and the generator through A will be tangent to the surface at A. Moreover, K is the pole of this plane, because KA and the generator through K are both lines of the complex, hence K is a point of the complex curve, and the points in which the plane α cuts this curve are the points of contact of the tangents drawn from the pole A of α to the curve of the section which α cuts from the surface S. Now when S is algebraic it follows that: *The degree of the complex curve is the same as the class of any plane section of the surface.*

It was shown by Clebsch* that all the asymptotic lines of a ruled surface can be found by means of integrations, provided one such line be given. It was shown by Bonnet† that the differential equation of the asymptotic lines of ruled surfaces is a Riccati equation; this equation possesses the property that the anharmonic ratio of four solutions is constant (theorem of Paul Serret). Hence, if three such asymptotic lines are known, all the others can be found without an integration. As the complex curve cuts each generator twice, it counts for two asymptotic lines; an integration will determine a third hence: *All the asymptotic lines of a ruled surface contained in a linear complex can be determined by a single integration.*

The complex curve of an algebraic ruled surface is itself algebraic, but the curves resulting from the integration of an algebraic function are usually transcendental, and in that case all the asymptotic lines except the complex curves are transcendental.

By means of these theorems Picard (*l. c.*) succeeded in determining all the ruled surfaces having algebraic asymptotic lines. Let Q be any algebraic curve, and C a linear complex. Through each point of q and in its osculating plane π lies one line l of C. This line will generate a ruled surface which is contained in C and has q for an asymptotic line; for, let π be the tangent plane to the surface at P on l; the tangents to the curve lie on the surface, hence the osculating plane of every point of q is tangent plane to the surface, *i. e.*, q is an asymptotic line. The tangents to q do

* "Ueber die Curven der Haupttangenten bei windschiefen Flächen," *Crelle's Journal*, vol. 68.

† "Théorie générale des surfaces," *Journal de l'École Polytechnique*, vol. 32.

not belong to C, and since S is contained in C, there are two more asymptotic lines determined as above. The surface now has three algebraic asymptotic lines; hence, from the general property of Riccati's equation which the asymptotic lines of the ruled surface satisfy, all the others are algebraic, and can be at once expressed without an integration.

Belonging to a given curve q there are ∞^5 surfaces which have algebraic asymptotic lines, and which touch each other along q. All ruled surfaces having algebraic asymptotic lines can be determined in this way.

Analytically, one may proceed as follows: let a curve q be defined in terms of a variable μ,

$$x = \varphi_1(\mu), \quad y = \varphi_2(\mu), \quad z = \varphi_3(\mu), \quad w = \varphi_4(\mu);$$

its osculating plane at a given point x, y, z, w is defined by the equation

$$\big(x, \varphi_2(\mu), \varphi_3'(\mu), \varphi_4''(\mu)\big) = 0.$$

The polar plane of the same point with regard to a given complex C is of the form

$$x(A\varphi_1 + B\varphi_2 + \cdots) + y(\cdots) + z(\cdots) + w(\cdots) = 0.$$

These two equations define a line of the complex in the pencil (μ,π). The ruled surface is the μ eliminant of these two equations. The succeeding steps are similar to those taken in the preceding case.

Application to Surfaces Having Two Rectilinear Directrices.

(a) *Directrices Distinct.*—Let $x = 0$, $y = 0$ be an m-fold line, and $z = 0$, $w = 0$ an n-fold line of a ruled surface; the equation of the surface will then be the expression of an $[m,n]$ correspondence between $\frac{z}{w}$ and $\frac{y}{x}$, namely:

$$(1) \qquad \varphi \equiv z^n\,{}_1u + z^{n-1}w\,{}_2u + \cdots + w^n\,{}_{n+1}u = 0,$$

in which ${}_ku$ is a binary quantic in x,y of degree m.

This surface is contained in the congruence whose directrices are x, y and z, w, which is expressed by the pencil of coaxial complexes

$$(2) \qquad xy' - x'y = k(zw' - z'w),$$

and consequently all the asymptotic lines are algebraic, and the inflexional tangents are distributed among the complexes of this pencil.

The tangent plane to (1) at the point (x', y', z', w') can be written in the form

$$(3) \qquad x'\frac{\partial\varphi}{\partial x'}\cdot\frac{x}{x'} + y'\frac{\partial\varphi}{\partial y'}\cdot\frac{y}{y'} + z'\frac{\partial\varphi}{\partial z'}\cdot\frac{z}{z'} + w'\frac{\partial\varphi}{\partial w'}\cdot\frac{w}{w'} = 0\,;$$

but
$$x'\frac{\partial\varphi}{\partial x'} + y'\frac{\partial\varphi}{\partial y'} = 0, \quad z'\frac{\partial\varphi}{\partial z'} + w'\frac{\partial\varphi}{\partial w'} = 0,$$

$$(4), \text{ hence} \quad w'\frac{\partial\varphi}{\partial x'}\cdot(xy' - x'y) + y'\frac{\partial\varphi}{\partial z'}(zw' - z'w) = 0.$$

Eliminating $xy' - x'y$ and $zw' - z'w$ between (1) and (4) there results

$$(5) \qquad y\frac{\partial\varphi}{\partial z} + kw\frac{\partial\varphi}{\partial x} = 0,$$

which, together with (1), defines the asymptotic lines.

Consider the plane $\lambda x = y$ through the m-fold line; substitute this value of y in (5), and let $\mu w = z$, then

$$(6) \qquad \lambda x^2\frac{\partial\varphi}{\partial z}(\lambda, \mu) + kw^2\frac{\partial\varphi}{\partial x}(\lambda, \mu) = 0.$$

From (6) the following properties can be immediately deduced:

I. When $\varphi = 0$, $\dfrac{\partial\varphi}{\partial x} = 0$, then $x = 0$ for all values of k; but these equations define those values of $\dfrac{z}{w}$ for which two generators issuing from the same point of the other multiple line coincide, *i. e.*, a *pinch point*, on the line $x = 0$, $y = 0$. Similarly $w = 0$ when $\dfrac{\partial\varphi}{\partial z} = 0$, $\varphi = 0$, which define the pinch points on the other multiple line; hence: *Every asymptotic line passes through all of the pinch points.*

II. Equation (6) together with $x = 0$, $w = 0$ defines an involution; hence: *The points of intersection of the asymptotic lines define an involution on every generator, with the points in which it cuts the multiple lines for foci.*

III. By changing the sign of k, real and imaginary points of intersection of an asymptotic line with a given generator interchange; hence: *No generator is cut by asymptotic lines belonging to both positive and negative complexes.*

It follows from these theorems that the cuspidal generators at the pinch points are inflexional tangents to all the asymptotic lines, as the two points of intersection approach the pinch point symmetrically when x approaches the pinch point.

The degree of these curves is, by Picard's theorem, $2mn - 2\delta$, in which δ is the number of double generators; it can also be obtained by counting the points of intersec-

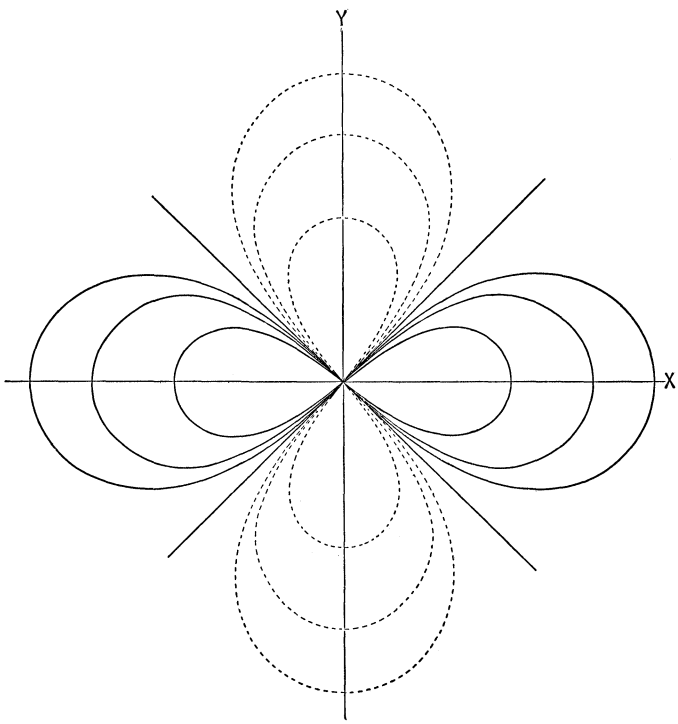

FIG. 1.

tion of any given curve with the plane $\lambda x = y$. This plane passes through the $2n(m-1)$ pinch points on the line xy and each of the n generators contained in the plane has two further points of intersection with the curve, but every double generator diminishes the number of pinch points by 2, hence the number of points is $2mn - 2\delta$, as before. In particular, if $\delta = (n-1)(m-1)$, the surface becomes unicursal and the order of the curves reduces to $2m + 2n - 2$, as was obtained by Cremona.* The method employed by Cremona will only apply when the surface is unicursal.

No modification is to be made for double generators; the corresponding factors will always divide out.

For the cubic surface [2, 1] the asymptotic lines are of order 4, and pass through the two pinch points on the double line. When the coördinate axes are rectangular

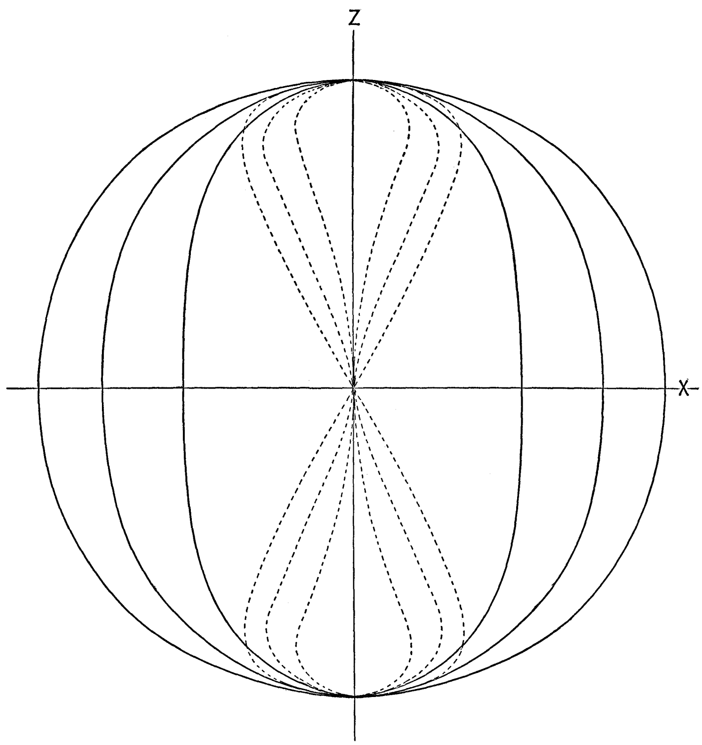

FIG. 2.

and the pinch points real and finite, the projection of the curves on the x, y and x, z planes are shown in Figs. 1 and 2. The dotted lines refer to left hand complexes.

For the quartic [2, 2] without a double generator [Cayley, I] the curves are of degree 8, and they pass through 8

* "Rappresentazione di una classe di superficie gobbe sopra un piano, e determinazione delle loro curve assintotiche." *Annali di matematica*, ser. 2, vol. 1 (1867–8).

pinch points, 4 on each double directrix. When the axes are rectangular, the cuspidal generators passing through the pinch points on the line z, w become asymptotes. When the pinch points are all real and distinct, a right hand curve and a left hand curve are shown in Fig. 3 and Fig. 4.

When the form [2, 2] has a double generator (Cayley, II) the asymptotic curves are of degree 6. There are now two pinch points on each line; this requires that, *e. g.*,

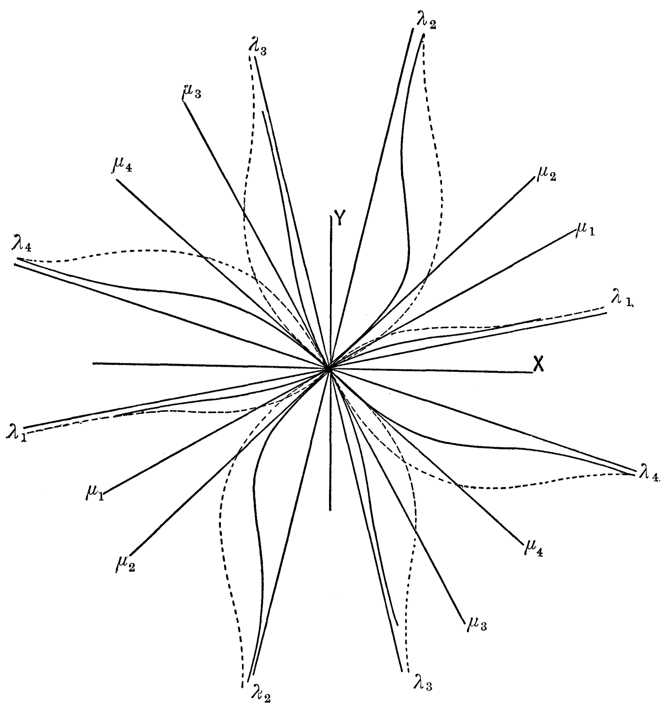

FIG. 3.

μ_2, μ_3, λ_2, λ_3 in Fig. 3, Fig. 4 all coincide. This form is unicursal. The form [3, 1] (Cayley, IX) is unicursal; it is of degree 6.

(β) *Directrices Coincident.*—The above method is not sufficient when the two directrices coincide. Suppose $n \leqq m$; from each point of the line issue n generators, all of which lie in a plane passing through the line. The points on the multiple line and the planes containing the line and the generators issuing from the point are projective. Hence in this case also the surface belongs to a linear congruence, but the congruence is now special. It is necessary to have a pencil of complexes having the multiple line for directrix, and having the same homography as the given surface. Let the line x, y be the multiple directrix, then

$$xz' - x'z - p(yw' - y'w) = 0$$

is a complex containing the line; p is to be so determined

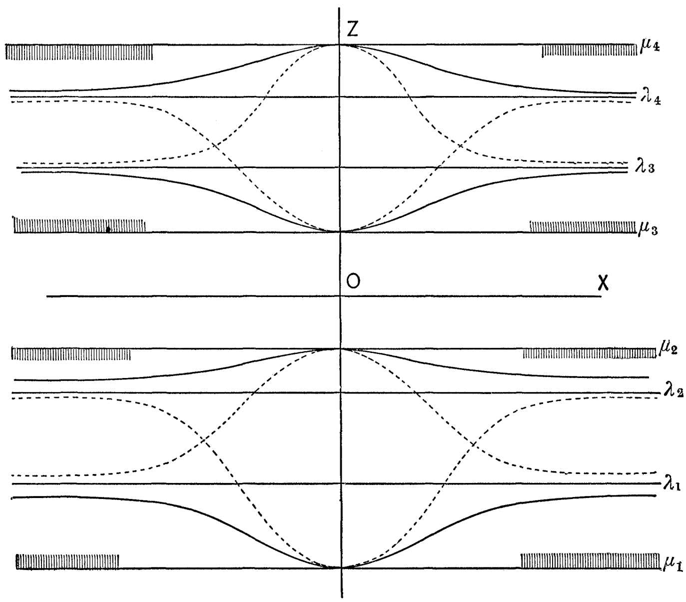

FIG. 4.

that as the point moves along the x, y axis, its polar plane will always coincide with the plane containing the corresponding generators of the surface. Finally $xy' - x'y = 0$ is a special complex with the multiple directrix for axis. By taking $p = 1$, the pencil of the complexes has the form

$$xz' - x'z - (yw' - y'w) + k(xy' - x'y) = 0$$

or, re-arranged,

$$(7)\qquad \begin{aligned}&(kx' + w')\,(yx' - y'x)\\ &+ [x(y'w' - x'z') - x'\,(y'w - x'z)] = 0.\end{aligned}$$

The equation of the surface now has the form

$$\varphi \equiv \sum_{r=0}^{n} v_2^{\,r} u_{m+n-2r} = 0$$

in which u_s is a binary quantic in x, y of degree s, and

$$v_2 \equiv yw - xz.$$

It is necessary to express the equation of the tangent plane to (8) at the point x', y', z', w' in the form

$$L(yx' - y'x) + M[x(y'w' - x'z') - x'(y'w - x'z)] = 0.$$

Since $\displaystyle x'\frac{\partial\varphi}{\partial x'} + y'\frac{\partial\varphi}{\partial y'} = \sum_{r=0}^{n} v_2^{\,r}(m+n-2r)u_{m+n-2r}$

$$+ \sum_{r=0}^{n} r v_2^{\,r} u_{m+n-2r} = (m+n)\sum_{r=0}^{n} v_2^{\,r} u_{m+n-2r} - \sum_{r=0}^{n} r v_2^{\,r} u_{m+n-2r},$$

we have

$$x'\frac{\partial\varphi}{\partial x'} + y'\frac{\partial\varphi}{\partial y'} = -\sum_{r=0}^{n} r v_2^{\,r} u_{m+n-2r} = -\left(z'\frac{\partial\varphi}{\partial z'} + w'\frac{\partial\varphi}{\partial w'}\right).$$

The equation of the tangent plane can now be put in the form

$$y'\frac{\partial\varphi}{\partial y'}\left(\frac{y}{y'} - \frac{x}{x'}\right) + w'\frac{\partial\varphi}{\partial w'}\left(\frac{w}{w'} - \frac{z}{z'}\right)$$

$$+ \sum_{r=0}^{n} r v_2^{\,r} u_{m+n-2r}\left(\frac{z}{z'} - \frac{x}{x'}\right) = 0;$$

but $\displaystyle \frac{(y'w' - x'z')}{y'}\,\frac{\partial\varphi}{\partial w'} = \sum_{r=0}^{n} r v_2^{\,r} u_{m+n-2r},$

hence

$$(9)\qquad \begin{aligned}&y'\frac{\partial\varphi}{\partial y'}\,(yx' - y'x)\\ &-\frac{\partial\varphi}{\partial w'}\,[x(y'w' - x'z') - x'(y'w - x'z)] = 0.\end{aligned}$$

The eliminant of (9) and (7) is, after a slight reduction,

$$\frac{\partial \varphi}{\partial y} + (kx + w)\frac{\partial \varphi}{\partial v_2} = 0 \tag{10}$$

which together with (8) defines the asymptotic lines.

Of the properties of these lines, II no longer applies, as the double lines coincide. The complexes are no longer coaxial, and the sign of the complex does not change with that of k; it is evident from equation (10) that every generator is cut but once (except at its intersection with the multiple line) by every asymptotic line. The property I, however, still holds, as may be proved as follows: All of the pinch points are now on the single multiple line, and there are $2m(n-1)$ of them. The direction of the plane $\lambda x = y$ is uniquely determined for each one of these points; the corresponding values of λ are determined from the vanishing of the discriminant of the binary equation

$$\varphi(x, \lambda w - z) = 0$$

or may be defined as those values which satisfy

$$\varphi = 0, \frac{\partial \varphi}{\partial v_2} = 0.$$

From equation (10), by replacing y by λx, it is seen that $\frac{\partial \varphi}{\partial v_2} = 0$ when $x = 0$, which proves that all the asymptotic lines pass through the pinch points, or it can be shown by eliminating z between (10) and (8); the result is the projection of the lines on the x, y, w plane. The coefficient of the highest power of w, equated to zero, will define the directions of the curves at the vertex $x = 0$, $y = 0$; these directions are independent of k and are easily shown to be the roots of the v_2 discriminant of φ.

To obtain the order of the curves in this case, the same method as before will suffice, when it is observed that the multiple line is itself a generator of multiplicity $m - n$. In (8) if λx be substituted for y, the directions of the n generators through the point $x = 0$, $z = \lambda w$ are given by the equation

$$u_{m+n}(\lambda) \cdot x^n + \cdots + (\lambda w - z)^n u_{m-n}(\lambda) = 0,$$

and one line will be $x = 0$ for every value of λ that satisfies $u_{m-n}(\lambda) = 0$; at these points one of the generators coincides with the multiple line. There are $2m(n-1)$ pinch points and these $m - n$ new points must count twice, for the asymptotes there touch the multiple line ; finally, every one of the n generators is cut by the same asymptotic line in but one point not on the directrix; the whole order is

$$2m(n-1) + 2(m-n) + n - 2\delta = 2mn - n - 2\delta.$$

In particular, if the surface is unicursal, the order reduces to $2m + n - 2$.

For the surface $\overline{[2, 2]}$ without double line (Cayley, IV) the order is 6, with a double line (Cayley, V) the order is 4. For these surfaces the multiple line cannot be a generator.

The surface $\overline{[3, 1]}$ (Cayley, VI) is unicursal ; its asymptotic lines are of order 5. The triple line is double generator.

For the cubic $\overline{[2, 1]}$ (Cayley's cubic scroll) the order is 3 ; the double directrix counts as single generator.

CORNELL UNIVERSITY,
November, 1898.

WILLSON'S GRAPHICS.

Theoretical and Practical Graphics. By F. N. WILLSON, C.E., M.A., Professor of Graphics, Princeton University. New York, The Macmillan Company, 1898. 4to, 300 pp. Price, $4.00.

THIS work, primarily, is a text book on graphics compiled by an experienced teacher to meet the needs of his own classes. Few student have, heretofore, been called upon to make larger expenditures for books than has the embryo engineer and in combining, under the comprehensive title of graphics, much that is essential for such students, for example, chapters upon freehand and mechanical drawing, theory of the helix, link motion, trochoidal and other mechanical curves and the theory of descriptive geometry, a real need has been recognized. This volume is far more than a collection of class room notes. Every page bears evidence of conscientious care and research. The grouping of the chapters, the concise and useful table of contents, the clear cut and often elaborate illustrations and the exceptional typographical excellence cannot be too highly commended.