THE APRIL MEETING OF THE CHICAGO SECTION.

THE first formal gathering of members of the AMERICAN MATHEMATICAL SOCIETY outside the places appointed for regular meetings took place at the University of Chicago in December, 1896. At this conference steps were taken to organize a section of the Society, which should have no independent interests but which should hold frequent meetings for the presentation of mathematical papers and the discussion of mathematical topics. Accordingly, the by-laws of the Society having been enlarged so as to admit of the formation of sections at convenient points, the Chicago Section was duly organized at a second conference held in April, 1897. Since then meetings have been held twice a year at regular intervals, and much benefit from a scientific point of view has accrued to those who have been so situated as to be able to attend.

The fifth regular meeting of the Section was held at Northwestern University, Evanston, on Saturday, April 1, 1899. The total attendance was twenty-four including the following members of the Society :

Professor Oskar Bolza, Professor E. W. Davis, Professor L. W. Dowling, Dr. Harris Hancock, Professor A. S. Hathaway, Professor Thomas F. Holgate, Mr. H. G. Keppel, Dr. Kurt Laves, Professor H. Maschke, Professor Malcolm McNeill, Professor E. H. Moore, Professor J. B. Shaw, Dr. H. F. Stecker, Professor H. S. White, Professor C. B. Williams, Professor J. W. A. Young.

A morning and an afternoon session were held, opening at 10:30 and 2:30 o'clock respectively. The First Vice-President of the Society, Professor E. H. Moore, occupied the chair. During the interval between the sessions the members took lunch together, the hour thus spent in social intercourse adding much to the interest of the meeting.

The following papers were read :

(1) Dr. HARRIS HANCOCK : "Primary functions."

(2) Professor E. W. DAVIS : "The group of the trigonometric functions."

(3) Professor H. MASCHKE : "On the continuation of a power series."

(4) Dr. KURT LAVES : "Lagrange's differential equations for a solid of variable form derived from Hamilton's principle."

(5) Professor E. H. MOORE : "The decomposition of

modular systems connected with the doubly generalized Fermat theorem (second communication)."

(6) Professor JAMES B. SHAW: "Some generalizations in multiple algebra and matrices."

(7) Professor J. W. A. YOUNG: "On the first presentations of the fundamental principles of the calculus."

(8) Professor A. S. HATHAWAY: "A new method of presenting the principles of the calculus."

(9) Professor E. H. MOORE: "On the subgroups of abelian groups."

(10) Mr. CARL C. ENGBERG: "A modification of the theory of the characteristics of evolutes (preliminary communication)."

(11) Dr. L. E. DICKSON: "Certain universal invariants of linear modular groups."

(12) Dr. L. E. DICKSON: "Concerning the four known simple groups of order 25920."

In the absence of the authors the paper by Mr. Engberg was presented by Professor Davis and those by Dr. Dickson were read by Professor Moore. Abstracts of the papers are given below.

Dr. Hancock's paper was an introduction to certain investigations undertaken by the author in Kronecker's modular systems. After a short account of the realm of rationality, modular, ideal, and prime ideal, the congruence

$$x^{p^f} - x \equiv 0 (\bmod \mathfrak{p}),$$

was discussed, where f is the degree of the prime ideal $\mathfrak{p}$, and p is the smallest prime integer that is divisible by $\mathfrak{p}$. It was shown that, if h is a divisor of f, $x^{p^h} - x \equiv \prod P(x) (\bmod p)$ where $P(x)$ is an integral prime function whose degree is equal to h or a divisor of h, and where the product is taken over all such divisors. The coefficient of the highest power of x in these functions is unity. Such functions are called *primary*. The number of primary functions that occur in the above product is denoted by X_h. It follows at once that $p^h = \sum dX_d$, where the summation is taken over all numbers d that are divisors of h. The quantities $X_1, X_2, \cdots, X_h, \cdots$ may be found by equating the coefficients of like powers of x in the identity

$$1 - px \equiv \prod_{d=1}^{d=\infty} (1 - x^d)^{\frac{X_1}{d}}.$$

In Professor Davis's paper it was pointed out that the expressions $\sin^2\theta$, $-\tan^2\theta$, $\sec^2\theta$, $\cos^2\theta$, $-\cot^2\theta$, $\operatorname{cosec}^2\theta$ can all be obtained from any one by the operations of subtraction from unity and reciprocation. Call these operations S and D, respectively. Then the trigonometric expressions above have the group generated by S, D. In fact, the six expressions are six cross ratios. Instead of the operations S and D we can use the substitution group generated by (x, y) and (x, r), the trigonometric functions having the usual definition in terms of x, y, and r. This substitution group is isomorphic with the operation group, but to an element of the one group does not always correspond the same element of the other. For example, to (x, y) which changes each function into its co-function, corresponds S, D, or SDS, according as the function operated on is $\sin\theta$, $\tan\theta$, or $\sec\theta$. If θ be the gudermannian of u, so that $\sin^2\theta = -\tan^2 iu$ with similar changes for the other expressions, then the substitution (y, r) changes the functions of iu into the complementary functions of iu, while if iv be complementary to iu the substitution (x, r) will likewise change functions of iv into complementary functions.

Professor Maschke's paper contained some simple and exceedingly interesting geometrical investigations on the continuation of a power series. The paper will be offered for publication to the *American Journal of Mathematics*.

The paper by Dr. Laves, the character of which is clearly indicated in the title, will be published in the *Astronomical Journal*.

In Professor Moore's first paper the theorem deduced was of the nature of a converse of the Fermat theorem, and in the pure arithmetic of the realm of rational integral functions of n indeterminates $y_1, \cdots, y_n$ with indeterminate coefficients. The paper will probably be published in the *Mathematische Annalen*.

Professor Shaw's paper presented the following considerations: (1) Hamilton's S may be made to apply to a field of order n_1, the units being $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, by defining it thus

$$S\,.\,\bar{\varepsilon}_i(x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_i\varepsilon_i + \cdots + x_n\varepsilon_n) = x_i\,;$$

(2) the Peirce vid $\varepsilon_i : \varepsilon_j$ is represented thus $\varepsilon_i S\bar{\varepsilon}_j$; (3) a matrix is an operator of the form $\sum_{ij}\varepsilon_i \cdot S\bar{\varepsilon}_j$; (4) if $\alpha = \sum a_i\varepsilon_i$, $\beta = \sum b_i\varepsilon_i$,

then $S\bar{a}\beta = S\bar{\beta}a = \sum a_i b_i$; (5) we may define $A\rho_1\rho_2 \cdots \rho_{n-1}$ by the form

$$\begin{vmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \cdots & \varepsilon_n \\ x_1' & x_2' & x_3' & \cdots & x_n' \\ x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ x_1^{(n-1)} & x_2^{(n-1)} & x_3^{(n-1)} & \cdots & x_n^{(n-1)} \end{vmatrix}$$

thus rendering several quaternion formulas susceptible of generalization for multiple algebra; and finally (6) by a process exactly that of Hamilton's for the linear vector operator of quaternions we may deduce the characteristic equation of a matrix. The paper is intended for publication in the *American Journal of Mathematics.*

Professor Young presented in synoptic form the ideas of Leibnitz and Newton concerning the foundation principles ofthe calculus, with casual references to some of the objections made at the time to their reasoning.

Professor Hathaway stated briefly the objections to each of the usual modes of presenting the calculus and then proceeded to outline a method which is essentially Newton's original method of treatment as presented by Hamilton in his Quaternions, adapted to modern difference notation, and which gives a process of differentiation that is more general than derivatives or rates, since it may be applied to functions of one, two, or more variables with equal facility and to functions that have no derivatives or rates.

It was shown by Kronecker that every abelian group A of finite order may be exhibited generationally by means of a system of generators $A_i (i = 1, \cdots, r)$ of respective periods a_i and by pairs commutative. The generators may be chosen so that the periods a_i are powers of primes, or say *primary.* To obtain a convenient form at once applicable to all abelian groups A of finite order, Professor Moore in his second paper (No. 9) introduces the generators $A_{ij} (i, j = 1, 2, 3, \cdots)$ of the respective periods $p_i^{a_{ij}}$ and by pairs commutative, where $p_i (i = 1, 2, 3, \cdots)$ are the successive primes $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$ and the $a_{ij} (i, j = 1, 2, 3, \cdots)$ are integers positive or zero, but in all only a finite number being positive and such that $a_{ij} \geqq a_{ij'}$ if $j < j'$. He calls the

matrix $\qquad (a_{ij}) \qquad (i, j = 1, 2, 3, \cdots)$

a primary basal character of the abelian group A. This character obviously determines the (abstract) abelian group A uniquely. He then proves the following theorem :

The necessary and sufficient conditions that a character (b_{ij}) $(i, j = 1, 2, 3, \cdots)$, where the b_{ij} are integers positive or zero such that $b_{ij} \geqq b_{ij'}$ if $j < j'$, be a primary basal character of a subgroup B of the abelian group A having a primary basal character (a_{ij}) $(i, j = 1, 2, 3, \cdots)$, are

$$b_{ij} \leqq a_{ij} \qquad (i, j = 1, 2, 3, \cdots).$$

To this theorem is attached the corollary : The group A has but one primary basal character. The corollary is the Frobenius and Stickelberger theorem of the primary invariants of the abelian group A, while the theorem is as to content the correction of an incorrect one stated by Burnside in §43 of his treatise.

Mr. Engberg pointed out in his preliminary communication that Salmon's theory of the characteristics of evolutes fails when the curve has multiple points at the circular points at infinity. Cayley worked out a complete theory for unicursal curves (Collected Works, vol. 8), but this had not been done for curves in general. The paper presented was an attempt to supply this deficiency ; when completed it will be offered for publication to the *American Journal of Mathematics.*

Dr. Dickson's first paper stated that the author had found two essentially different proofs of the following general theorem : The group of all linear homogeneous substitutions on m indices taken modulo 2 leaves invariant the function $s_1 + s_2 + s_3 + \cdots + s_m$, where s_r denotes the elementary symmetric function $\sum x_{i_1} x_{i_2} \cdots x_{i_r}$, summed for every combination of the m indices x_i taken r together. The following theorem is verified immediately, viz. : Every linear homogeneous substitution of determinant D on m indices with coefficients belonging to the $GF[p^n]$ multiplies by D the following relative invariant :

$$\begin{vmatrix} x_1 & x_2 & \cdots x_m \\ x_1^{p^n} & x_2^{p^n} & \cdots x_m^{p^n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_1^{p^{n(m-1)}} & x_2^{p^{n(m-1)}} & \cdots x_m^{p^{n(m-1)}} \end{vmatrix}.$$

In Dr. Dickson's second paper (No. 12), which is intended for publication in the *Proceedings of the London Mathematical Society*, an abstract group H is constructed, of order 25920, simply isomorphic with each of the four simple groups obtained by the decomposition of the following four linear homogeneous groups :

(1) The abelian group on four indices modulo 3.

(2) The second hypoabelian group on six indices modulo 2.

(3) The orthogonal group on five indices modulo 3.

(4) The hyperabelian group on four indices in the $GF[2^2]$.

The abstract group H is generated by six operators subject to the complete set of generational relations

$$(a)\quad \begin{cases} B_1^2 = B_2^2 = B_3^2 = B_4^2 = B_5^2 = 1, \\ (B_1B_2)^3 = (B_2B_3)^3 = (B_3B_4)^3 = (B_4B_5)^3 = 1, \\ (B_1B_3)^2 = (B_1B_4)^2 = (B_1B_5)^2 = (B_2B_4)^2 = (B_2B_5)^2 \\ \qquad = (B_3B_5)^2 = 1; \end{cases}$$

$$(b)\quad \begin{cases} B^3 = 1,\ BB_1 = B_1B,\ BB_2 = B_2B,\ BB_4 = B_4B, \\ BB_5 = B_5B,\ (BB_3)^3 = B_1B_5, \\ B(B_1B_5B_3B_2B_4)^3B = (B_1B_5B_3B_2B_4)^3, \\ (B_1B_2B_3B_1B_2)(B^{-1}B_3B)(B_1B_2B_3B_1B_2)(B^{-1}B_3B) = B_5. \end{cases}$$

The operators B_1, B_2, B_3, B_4, B_5, subject to the relations (a), generate a subgroup simply isomorphic with the symmetric group on six letters (Moore's Theorem A, "Concerning the abstract groups of order k !" etc., *Proceedings of the London Mathematical Society*, vol. 28, no. 597).

The group H is proven simply isomorphic with the substitution group of order 25920 on 36 letters generated by the following three substitutions, the first two of which generate a subgroup simply isomorphic with the symmetric group on six letters (Moore's Theorem A', l. c.) :

$[B_1]$: (5, 6) (9, 10) (11, 12) (14, 15) (18, 19) (20, 21) (23, 24) (25, 26) (29, 30) (31, 32) (33, 34) (35, 36).

$[C]$: (2, 6, 17, 3, 15, 8) (4, 10, 16, 20, 14, 11) (5, 21, 13, 19, 7, 12) (23, 34, 27, 26, 35, 30) (9, 18) (22, 29) (24, 36, 28) (25, 32, 33).

$[B]$: (1, 2, 3) (4, 13, 22) (5, 14, 23) (6, 15, 24) (7, 16, 27) (8, 17, 28) (9, 18, 25) (10, 19, 26) (11, 21, 30) (12, 20, 29).

The next meeting of the Section will be held at the University of Chicago on Thursday and Friday, December 28 and 29, 1899.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

AN ELEMENTARY PROOF THAT BESSEL'S FUNCTIONS OF THE ZEROTH ORDER HAVE AN INFINITE NUMBER OF REAL ROOTS.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of February 25, 1899.)

THE only elementary proof * with which I am acquainted that the function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

has an infinite number of real roots is the one originally given by Bessel (cf. Gray and Mathews : Treatise on Bessel Functions p. 44). I wish to call attention to a second elementary method of proving this theorem. Although this method is tolerably obvious I do not think it has been used for this purpose before.

In the first place, it is clear from the series for $J_0(x)$ that this function has at least one positive root ; for if we substitute in this series first the value $x = 0$, and then the value $x = 3$, we get first a positive and then a negative value. Let us denote the smallest positive root of $J_0(x)$ by c, a quantity whose value can be readily computed as $2.405\cdots$.

We will now prove the theorem :

Any real solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \tag{1}$$

has an infinite number of real roots.

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* The proofs frequently met with, one depending on the asymptotic value of $J_0(x)$, and the other on what I have called (cf. BULLETIN, vol. 4, p. 298) Sturm's theorem of comparison, cannot be regarded as elementary as they depend on general theorems which can hardly be proved rigorously without some rather delicate analysis.