The next meeting of the Section will be held at the University of Chicago on Thursday and Friday, December 28 and 29, 1899.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

AN ELEMENTARY PROOF THAT BESSEL'S FUNCTIONS OF THE ZEROTH ORDER HAVE AN INFINITE NUMBER OF REAL ROOTS.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of February 25, 1899.)

THE only elementary proof* with which I am acquainted that the function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

has an infinite number of real roots is the one originally given by Bessel (cf. Gray and Mathews : Treatise on Bessel Functions p. 44). I wish to call attention to a second elementary method of proving this theorem. Although this method is tolerably obvious I do not think it has been used for this purpose before.

In the first place, it is clear from the series for $J_0(x)$ that this function has at least one positive root; for if we substitute in this series first the value $x = 0$, and then the value $x = 3$, we get first a positive and then a negative value. Let us denote the smallest positive root of $J_0(x)$ by c, a quantity whose value can be readily computed as $2.405\cdots$.

We will now prove the theorem :

Any real solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \tag{1}$$

has an infinite number of real roots.

*The proofs frequently met with, one depending on the asymptotic value of $J_0(x)$, and the other on what I have called (cf. BULLETIN, vol. 4, p. 298) Sturm's theorem of comparison, cannot be regarded as elementary as they depend on general theorems which can hardly be proved rigorously without some rather delicate analysis.

Since $J_0(x)$ is a particular solution of (1) the theorem just stated includes as a special case the theorem we set out to prove. The proof of our theorem follows immediately from the following formula in which $F(x)$ denotes any real solution of (1), a a positive constant, and x a real quantity confined to the interval $a > x > -a$

$$(2) \qquad \pi F(a) J_0(x) = \int_0^\pi F\sqrt{a^2 + x^2 - 2ax\cos\varphi}\, d\varphi.$$

If in this formula we let $a > c$ and $x = c$, we get

$$\int_0^\pi F\sqrt{a^2 + c^2 - 2ac\cos\varphi}\, d\varphi = 0.$$

Now as φ varies from 0 to π the argument of F varies from $a - c$ to $a + c$. Therefore $F(x)$ is zero for some value of x between these limits, which it should be noticed are any two quantities either both positive or both negative and differing by $2c$. We have thus proved *that every real solution of* (1) *has at least one root in any interval of length* $2c = 4.810 \cdots$, *not including the origin, and therefore an infinite number of real roots.* Incidentally we have found an upper limit (not very close to be sure) for the difference between two successive roots.

Although formula (2) which we have here used is not unfamiliar* it is perhaps well in order to bring out its thoroughly elementary character to deduce it here from first principles. In doing this I use a method not essentially different from that used by Heine in his Handbuch der Kugelfunktionen, second edition, vol. 1, § 83.†

Letting $y = F\sqrt{a^2 + x^2 - 2ax\cos\varphi}$ we find by direct computation that the first member of (1) reduces to

$$\frac{\partial}{\partial\varphi}\left[\frac{-a\sin\varphi}{x\sqrt{a^2 + x^2 - 2ax\cos\varphi}}\; F'\sqrt{a^2 + x^2 - 2ax\cos\varphi}\right].$$

We see therefore that

$$\int_0^\pi F\sqrt{a^2 + x^2 - 2\,ax\cos\varphi}\;\, d\varphi$$

is, provided that $|x| \neq a$, a solution of (1). Moreover this integral does not become infinite when $x = 0$, and since

* It is not contained in Gray and Mathews's Treatise although the special case in which $F = J_0$ follows immediately from formula (69′) of that book.

† An entirely different method is given in § 84.

every solution of (1) linearly independent of $J_0(x)$ does become infinite when $x = 0$ we see that the above integral must be equal when $a > x > -a$ to a constant multiple of $J_0(x)$. By letting $x = 0$ we see that this constant has the value $\pi F(a)$.

I have arranged the above treatment so as to avoid any reference to partial differential equations. It is, however, very closely connected with the following theorem :

Any solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0 \tag{3}$$

*which together with its first and second partial derivatives is single valued and continuous throughout a certain region of the (x,y)-plane will change sign at two or more points of any circle of radius $c = 2.405 \cdots$ which lies wholly in this region.**

* Cf. H. Weber, *Math. Annalen*, vol. 1, p. 10, or for a less simple proof Pockels's book: "Ueber die partielle Differentialgleichung $\Delta u + k^2 u = 0$," p. 217.

I take this opportunity of referring to another point in Pockels's book (pp. 225–228). It was here, I believe, that the theorem was for the first time stated and proved that at a point at which n curves $u = 0$ cross each other (u being a solution of the equation $\Delta u + k^2 u = 0$) they make angles π/n with one another. The method of proof there used seems to me unsatisfactory from the point of view of rigor, at least in the general case in which k^2 is a function of (x, y). The following proof will, I think, be found rigorous and if possible even simpler than Pockels's proof. We will consider at once the more general differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \phi \frac{\partial u}{\partial x} + \chi \frac{\partial u}{\partial y} + \psi u = 0$$

where ϕ, χ, ψ are at the point in question (which for simplicity I take as the origin) analytic functions of (x, y). We know (Cf. Picard : *Journal de l'École Polytechnique*, Cahier 60, p. 91) that any solution u of the above equation which, together with its first and second partial derivatives, is continuous throughout a region including the origin will be analytic at the origin. Developing u by Maclaurin's theorem we get (since the curve $u = 0$ is to have a multiple point at the origin) $u = u_n + u_{n+1} + \cdots$ where u_k is a homogeneous polynomial of degree k. Substituting this in the differential equation we see at once that u_n satisfies Laplace's equation and, therefore, $u_n = 0$ represents n straight lines through the origin making angles π/n with one another, and these lines are the tangents to the curve $u = 0$ at the origin.

By the same method we see that if the curve $u = c$ (where c is a constant different from zero) has a multiple point of the nth order at P, the tangents at this point will make angles π/n with one another when and only when the curve $\psi = 0$ has a multiple point at P of order at least $n - 1$. (In order that $u = c$ should have a multiple point of the nth order at P it is necessary that $\psi = 0$ should have there a multiple point whose order is at least $n - 2$.)

It is hardly necessary to add that these same methods admit of application to the similar problems in space of three dimensions.

This theorem can also be immediately applied to Bessel's functions whose order is not zero. Let $F_n(x)$ be any real solution of Bessel's equation

$$(4) \qquad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0.$$

Using polar coördinates r, ϑ we have as a solution of (3) when n is real

$$u = \cos n\vartheta \cdot F_n(r),$$

when n is pure imaginary

$$u = e^{in\vartheta} \cdot F_n(r).$$

Applying the theorem just quoted to these solutions we get the theorems:

If $n^2 \leqq 1$, $F_n(x)$ *vanishes at least once in any interval of length* $2c = 4.810 \cdots$ *which does not include the origin.*

If $n > 1$, $F_n(x)$ *vanishes at least once in any interval of length* $2c$ *throughout which* $|x| > c \left[\csc\frac{\pi}{2n} - 1\right]$.

As a special application I note that we thus get an upper limit for the value of the smallest root of $F_n(x)$ and thus in particular of $J_n(x)$.

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A GENERALIZATION OF APPELL'S FACTORIAL FUNCTIONS.

BY DR. E. J. WILCZYNSKI.

(Read before the American Mathematical Society at the Annual Meeting, December 28, 1898.)

LET
$$F(s, z) = 0$$

be an algebraic equation defining s as function of z. Let R, the corresponding Riemann's surface, be of class p. By a system of crosscuts $a_1, \cdots, a_p$; $b_1, \cdots, b_p$; $c_1, \cdots, c_p$ the $(2p + 1)$-ply connected surface R is changed into a simply connected surface R_{abc}.