THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 28, 1899. Thirty persons were in attendance at the two sessions, including the following members of the Society:

Professor Maxime Bôcher, Professor E. W. Brown, Professor F. N. Cole, Professor T. W. Edmondson, Professor A. M. Ely, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Miss Carrie Hammerslough, Mr. S. A. Joffe, Mr. C. J. Keyser, Dr. G. H. Ling, Professor F. H. Loud, Dr. Emory McClintock, Dr. James Maclay, Dr. G. A. Miller, Professor E. H. Moore, Professor F. Morley, Professor B. O. Peirce, Professor A. W. Phillips, Professor James Pierpont, Professor M. I. Pupin, Miss Amy Rayson, Professor C. A. Scott, Professor H. D. Thompson, Professor E. B. Van Vleck, Professor A. G. Webster, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair. At the opening of the morning session a recess was taken to enable the members to attend the presidential address of Professor H. A. Rowland before the American Physical Society, which was in session in the same building. The simultaneous meeting of the two Societies, which is intended to be a permanent feature, naturally added to the interest and activity on both sides. The two organizations are modelled on a common plan; they have many members in common; and their hearty coöperation, which seems assured, cannot fail to be an important factor in the advancement of science.

The Council announced the election of the following persons to membership in the Society: Professor M. E. Bogarte, Northern Indiana Normal School, Valparaiso, Ind.; Mr. A. S. Gale, Yale University, New Haven, Conn.; Mr. B. F. Groat, University of Minnesota, Minneapolis, Minn.; Dr. Edward Kasner, Columbia University, New York, N. Y.; Professor J. A. Miller, University of Indiana, Bloomington, Ind.; Professor A. M. Sawin, Clark University, Atlanta, Ga.; Professor S. A. Singer, Capital University, Columbus, Ohio; Dr. H. E. Slaught, University of Chicago, Chicago, Ill.; Professor E. P. Thompson, Miami University, Oxford, Ohio. Seven applications for membership were received.

With a view to economy of time, the Council voted that hereafter the morning session of the meetings of the Society should open at 11 a. m. and the afternoon session at 2 p. m. The Council will meet at 10.15 a. m.

The next meeting of the Society, Thursday, December 28th, will be the annual meeting for the election of officers and three other members of the Council. At this meeting President R. S. Woodward will deliver a presidential address on "The century's progress in applied mathematics."

The following papers were presented at this meeting:

(1) Professor PAUL GORDAN: "Formentheoretische Entwickelung der in Herrn White's Abhandlung über Curven dritter Ordnung enthaltenen Sätze."

(2) Professor E. O. LOVETT: "The transformation of straight lines into spheres."

(3) Dr. G. A. MILLER: "On the simply transitive primitive groups."

(4) Professor C. A. SCOTT: "The conditions imposed on a curve by assigned multiple points."

(5) Professor E. H. MOORE: "On the generational determination of abstract groups" (preliminary communication).

(6) Professor C. A. SCOTT: "The status of imaginaries in pure geometry."

(7) Professor MAXIME BÔCHER: "On Sturm's theorem of comparison" (preliminary communication).

(8) Professor F. MORLEY: "On a fundamental geometric construction."

(9) Mr. E. B. WILSON: "The decomposition of the general collineation of space into three skew reflections."

(10) Dr. G. A. MILLER: On the order of the product of two substitutions."

(11) Mr. J. K. WHITTEMORE: "On a generalization of the fundamental problem of the calculus of variations."

(12) Mr. J. L. COOLIDGE: "A projective representation of the imaginary points of a plane."

Professor Gordan's paper was presented to the Society through Professor H. S. White, and Mr. Coolidge's through Professor Bôcher. Mr. Wilson was introduced by Professor Pierpont. In the absence of the authors the paper of Professor Gordan was read by Professor Moore, that of Professor Lovett was read by title, and those of Mr. Whittemore and Mr. Coolidge were read by Professor Bôcher.

Dr. Miller's first paper appears in the present number of the BULLETIN. Miss Scott's second paper will appear in a subsequent number. Abstracts of the other papers are given below.

Professor Gordan's paper will appear in the *Transactions.* The following is a summary of its content: The theorems proven by Professor H. S. White in a paper read at the Columbus meeting of this Society in August, 1899, were established by the use of a canonical form of the ternary cubic. These theorems can be obtained quite as readily without using a canonical form. For this purpose the Hessian and Cayleyan are written for the linear combinations $\kappa f + \lambda\triangle$ and $\kappa\pi + \lambda s$, where f denotes a nonsingular ternary cubic, $\triangle$ its Hessian, π its conjugate cubic in contragredient variables, and s its Cayleyan, the Hessian of π. The forms so found are cubics in κ, λ; hence are determined the parameters of curves having the same Hessian or the same Cayleyan as either f or π. These parameters contain a square root of an invariant of f, but no other irrationality. From explicit formulæ so found it is easy to calculate the simplest invariants of each of the eight forms called A, B, F, G, φ, ψ, α, β in the paper of Professor White, and to show that for certain pairs of them a simultaneous covariant resembling θ_f must be identical with one of the pair multiplied into an invariant factor. These invariant factors are explicitly determined; and the conjugate property of A and α, B and β, F and φ, G and ψ is exhibited. Finally it is shown that a set of three invariantive equations is sufficient to characterize completely such a triple system as f, A, B. The process is in general the converse of that employed in the earlier paper cited above, and confirms its results, with the addition of important details.

Professor Lovett's paper, which will appear in the *American Journal of Mathematics,* is in abstract as follows: The transformations of the point space (x, y, z) into the point space (X, Y, Z) which are determined by the two *æquationes directrices*

$$x\varphi_1 + y\varphi_2 + z\varphi_3 + \varphi_4 = 0, \quad x\varphi_5 + y\varphi_6 + z\varphi_7 + \varphi_8 = 0, \quad (1)$$

where $\varphi_i \equiv a_iX + b_iY + c_iZ + d_i$ (a_i, b_i, c_i, d_i constants),

define a family of ∞^{30} transformations which change the straight line

$$y + kx + m = 0, \quad z + lx + n = 0, \quad (2)$$

into the quadric

$$\begin{vmatrix} k & m \\ l & n \end{vmatrix} \cdot \begin{vmatrix} \varphi_2 & \varphi_3 \\ \varphi_6 & \varphi_7 \end{vmatrix} - k \begin{vmatrix} \varphi_2 & \varphi_4 \\ \varphi_6 & \varphi_8 \end{vmatrix} - l \begin{vmatrix} \varphi_3 & \varphi_4 \\ \varphi_7 & \varphi_8 \end{vmatrix} - m \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_5 & \varphi_6 \end{vmatrix}$$
$$- n \begin{vmatrix} \varphi_1 & \varphi_3 \\ \varphi_5 & \varphi_7 \end{vmatrix} + \begin{vmatrix} \varphi_1 & \varphi_4 \\ \varphi_5 & \varphi_8 \end{vmatrix} = 0. \quad (3)$$

This quadric reduces to a sphere, without k, l, m, n entering the equations of condition, in the following cases and in no other: 1° when any determinant of the matrix

$$\begin{Vmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \end{Vmatrix} \qquad (\psi_i \equiv a_i X + b_i Y + c_i Z \equiv \varphi_i - d_i), \quad (4)$$

reduces to the form $\quad$ Const. $(X^2 + Y^2 + Z^2)$, $\quad$ (5)

and at the same time the functions φ_i corresponding to the ψ remaining in the matrix reduce to constants; for example, $\varphi_2, \varphi_3, \varphi_6, \varphi_7$ constants and the determinant $\psi_1\psi_8 - \psi_4\psi_5$ of the form (5); 2° when all six determinants of the matrix (4) are of the form (5).

The first case gives six families each of ∞^{18} line-sphere contact transformations determined by two linear equations of the form

$$\begin{gathered}(a_1X + b_1Y + c_1Z + d_1)x + d_2y + d_3z + a_4X \\ + b_4Y + c_4Z + d_4 = 0, \\ (a_5X + b_5Y + c_5Z + d_5)x + d_6y + d_7z + a_8X \\ + b_8Y + c_8Z + d_8 = 0,\end{gathered} \quad (6)$$

where the constants are subject to the conditions

$$\begin{gathered}a_1a_8 - a_4a_5 = b_1b_8 - b_4b_5 = c_1c_8 - c_4c_5, \\ a_1b_8 + b_1a_8 - a_4b_5 - b_4a_5 = 0, \; b_1c_8 + c_1b_8 - b_4c_5 - c_4b_5 = 0, \\ c_1a_8 + a_1c_8 - c_4a_5 - a_4c_5 = 0.\end{gathered} \quad (7)$$

Assigning the following particular values to the above constants

$$\begin{gathered}a_1 = a_8 = b_1 = b_8 = c_4 = c_5 = d_1 = d_2 = d_4 \\ = d_5 = d_7 = d_8 = 0, \\ a_4 = a_5 = c_1 = - c_8 = d_3 = d_6 = 1, \\ b_4 = - b_5 = i = \sqrt{-1},\end{gathered} \quad (8)$$

we have the celebrated correspondence of Lie determined by the equations

$$Zx + z + X + iY = 0, \quad (X - iY)x + y - Z = 0. \quad (9)$$

The above transformations change right lines to points if

$$d_1 : d_2 : d_3 : d_4 = d_5 : d_6 : d_7 : d_8. \tag{10}$$

The second class of conditions demands that the functions ψ_i have the forms

$$\begin{aligned} \psi_j &\equiv a_j X + i a_j Y + c_j Z, \\ \psi_{j+4} &\equiv c_j X - i c_j Y - a_j Z \end{aligned} \quad (j = 1,2,3,4); \tag{11}$$

the corresponding equations (1) define a fifteen-parameter family of line-sphere contact transformations whose transformations may be derived by operating on the transformations of the ordinary projective group with the transformation (9) of Lie.

In conclusion the note points out the corresponding generalizations for spaces of higher dimensions which have been incorporated in another paper; sets up a six-parameter family of line-sphere contact transformations by generalizing the form given by Darboux* to Lie's transformation; and finally calls attention to the fact that infinite families† of line-sphere contact transformations have been constructed.

If the nominal conditions imposed on a curve by points of assigned order are in number N, and the actual conditions are in number D, the m-ic excess e_m is $N-D$. The variation in e_m for different values of m has been studied in detail by Dr. Macaulay (*Proceedings of the London Mathematical Society*) for the case where all given points are simple. The object of Professor Scott's paper was to give independent proofs for the case of multiple points. The theorems proved are: (1) if m be sufficiently great, $e_m = 0$; (2) $e_m - e_{m+1} \leq m$, where the sign of equality can occur only in a particular case; (3) $e_m - e_{m+1} \geq 0$, where the sign of equality occurs only if $e_m = 0$.

On the basis of the current definition of abstract groups (discontinuous or continuous, of order finite or infinite) Professor Moore's paper investigates under what conditions and in what sense a set of generator symbols subject to certain generational relations serves to determine a definite abstract group. This paper will be published in the *Transactions*.

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*Darboux, Théorie des surfaces, vol. 1, §157.

†*Comptes rendus*, August 21, 1899.

It was the object of Professor Bôcher's communication to show how some of Sturm's theorems concerning linear differential equations of the second order can be proved rigorously and at the same time simply. The analogy of a part of the method here used with a method suggested by Mr. Porter (Cf. BULLETIN, March, 1897, p. 210) should be noticed. Rigorous but complicated proofs had previously been given by the author in the BULLETIN for April, 1898.

Sturm's first "theorem of comparison"* will evidently be proved if it can be shown that y_2 has at least one root in each of the intervals $a<x<x_1$, $x_1<x<x_2$, $\cdots$. By combining equations numbered (1) and (2) we readily get

$$y_2(x_1)y_1'(x_1) = \int_a^{x_1}(\varphi_1 - \varphi_2)y_1y_2dx.$$

If y_2 had no root in the interval $a<x<x_1$, y_1 and y_2 would have the same sign throughout this interval. The right side of the above equation would therefore be greater than zero. The left side is, however, obviously either negative or zero. In either case we have a contradiction, and we thus see that y_2 has at least one root in the interval $a<x<x_1$. Consider now the solution $\overline{y}_2$ of (2) which vanishes at x_1 and whose derivative there has the same value as the derivative of y_1 at that point. By a proof exactly like the one just given we see that $\overline{y}_2$ has at least one root in the interval $x_1<x<x_2$, and, therefore, by a well known and easily proved theorem of Sturm,* the same is true of y_2. In the same way each of the other intervals $x_2<x<x_3$, $x_3<x<x_4$, $\cdots$ is treated. The second theorem of comparison (*loc. cit.*, p. 300) admits an analogous treatment; and the same is true of certain generalized forms of these theorems.

Professor Morley proved the following theorem: Given three points and four planes, take the polar of any two planes as to the points, and let this polar point be joined to the intersection of the other two planes. We thus have six planes. These meet in a point; and the third polar of this point as to the four planes is the polar of the original three points as to the original four planes. A precisely similar theorem holds for $n+1$ spaces S_{n-1} and n points in a space S_n of n dimensions. That is, the polar of any $n-1$ spaces

* Loc. cit., p. 298. The notation there explained is used here without change.

* Cf. BULLETIN, March, 1897, p. 209.

S_{n-1} as to the points is joined to the intersection of the other two S_{n-1}'s. We have thus $n(n+1)/2$ spaces S_{n-1} which meet in a point; the n^{th} polar of this point as to all the S_{n-1}'s is the polar of the n given points as to all the S_{n-1}'s. This yields, for points and lines lying in a plane, the proof of some statements made by Caporali (Works, Pellerano, Naples, 1888; pp. 262–269).

Mr. Wilson's paper showed that the collineation in three dimensional space can, in general, be resolved into three skew reflections. The method pursued was to give a geometrical construction for three skew reflections whose product was the given collineation. The paper is offered for publication in the *Transactions.*

Dr. Miller showed that if any three arbitrary numbers larger than unity are given we can always find three substitutions s_1, s_2, s_3 whose orders are these three numbers respectively, in the given order, and which satisfy the equation $s_1 s_2 = s_3$. These substitutions can always be so selected that the number of the different elements involved in all of them does not exceed the order of the largest of the given numbers increased by two. In the proof of this theorem the following lemma was frequently used: If two circular substitutions have an odd number of consecutive elements in common their product is a circular substitution that involves all the elements of both.

Mr. Whittemore's paper is in abstract as follows: The fundamental problem of the calculus of variations may be said to be: To determine y, a function of x, whose values for $x = a$ and $x = b$ are given, such that

$$u = \int_a^b F(x,y,y')dx \tag{1}$$

shall be a maximum or a minimum. In (1) F is a given function, supposed analytic, of x,y, and $y' = dy/dx$. This problem is identical with the following: To determine y, a function of x, whose values for $x = a$ and $x = b$ are given, such that a solution of the differential equation

$$u' = F(x,y,y') \tag{2}$$

which vanishes for $x = a$, shall have a maximum or a minimum for $x = b$. The second statement leads naturally to the posing of the same problem for the more general differential equation

$$u' = F(x,u,y,y'). \tag{3}$$

If we denote by η the variation of y and by π the corresponding "first variation" of u, and if we suppose η to be a "regular" variation, that is such a variation that η'^2 may be neglected as well as η^2, it follows from (3) that π is determined by the linear differential equation of the first order

$$\pi' - \frac{\partial F}{\partial u}\pi - \frac{\partial F}{\partial y}\eta - \frac{\partial F}{\partial y'}\eta' = 0, \tag{4}$$

where we have further the initial condition $\pi(a) = 0$. A necessary condition that u shall have for $x = b$ a maximum or a minimum is $\pi(b) = 0$ for all regular variations η where $\eta(a) = \eta(b) = 0$. If now in (4) we regard u, y, and η as known functions of x, we can find π by quadratures, and we may then see by the usual method of the calculus of variations that we must necessarily have

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{5}$$

This condition corresponds to Lagrange's necessary condition for a maximum or a minimum in the calculus of variations, and reduces to it when $\dfrac{\partial F}{\partial u} = 0$.

If now we consider the differential equation

$$u'' = F(x,u,u',y,y') \tag{6}$$

we have for π, the "first variation" of u, a linear differential equation of the second order

$$\pi'' - \frac{\partial F}{\partial u'}\pi' - \frac{\partial F}{\partial u}\pi - \frac{\partial F}{\partial y}\eta - \frac{\partial F}{\partial y'}\eta' = 0, \tag{7}$$

which cannot be solved by quadratures. A method is therefore necessary which does not depend upon the integration of (7). Such a method is discussed in the present paper, and applied to the equation (3) leads again to (5). Applied to (6) the method leads to the proposition: There is in general no function y, having given values for $x = a$ and $x = b$ such that u, determined by (6) and the two initial conditions $u(a) = u_0$, $u'(a) = u_0'$, shall have a maximum or a minimum for $x = b$.

It seems probable that the same method would lead to a similar statement for all differential equations of an order greater than one in u. The presence in the differential equations of derivatives of y of order greater than one does not affect the result in the latter case, while in the case of equation (3), as in the calculus of variations, we are led to a resulting differential equation more complicated than (5).

Mr. Coolidge's paper began with a definition of imaginary geometrical elements after von Staudt and August. The usual method of representing imaginary points in a real line by real points in a plane was then mentioned, with an explanation why this proceeding could not be so extended as to include all imaginary points in a plane. It was then shown that every imaginary point in a real plane might be represented by the real base of its projection from a real point on an imaginary plane, and the necessary construction was explained. The assemblages of lines representing points on real and imaginary lines were discussed, as well as the relations of those lines which represented chains of imaginary points. The paper closed with a brief reference to the congruences of lines representing points lying on coinc sections, and a discussion of the advisability of finding a method of representing all the imaginary points in space.

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NOTE ON THE SIMPLY TRANSITIVE PRIMITIVE GROUPS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, October 28, 1899.)

THE simply transitive primitive groups present many difficulties which have not yet been solved. In his "Traité des substitutions" Jordan proved some fundamental theorems in regard to these groups.* Several additional theorems appeared in the "*Proceedings of the London Mathematical Society*," 1897, pp. 534–536. The following theorems and corollaries are closely connected with the results of this article.

Every primitive group G of degree n contains a maximal subgroup G_1 of degree $n-1$. When G is simply transitive

* P. 284.