commutators of this holomorph. When this order is even, the commutator subgroup of the holomorph includes half of the operators of this cyclical group and all of these operators are commutators of this holomorph.

Since $s^{-1}t^{-1}$ is similar to ts and this is similar to st, we observe that the commutator of two operators is similar to the commutator formed by means of one of these operators and the inverse of the other. The preceding results are, in part, supplementary to those contained in the paper "On the commutator groups," BULLETIN, Vol. IV., pp. 135–139.

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THE CALCULUS OF GENERALIZATION.

Calcul de Généralisation. Par G. OLTRAMARE, Doyen de la Faculté des Sciences de l'Université de Genève. Paris, A. Hermann, 1899. 8vo, viii + 191 pp.

THIS work is the magnum opus of the venerable dean of the faculty of sciences, of Geneva, who is probably the oldest living pupil of Cauchy. The volume recapitulates and completes the works of the author published during the last twenty years.

Oltramare regards every function as developable in a series of exponentials; thus, a designating an independent variable, he puts

$$\varphi(a) = A_\alpha e^{\alpha a} + A_\beta e^{\beta a} + A_\gamma e^{\gamma a} + \cdots,$$

where $\alpha, \beta, \gamma, \cdots$ are any constants real or imaginary, in number finite or infinite. He adopts the shorter notation Ge^{au} for the series $\Sigma A_u e^{au}$, u taking successively the values $\alpha, \beta, \gamma, \cdots$, and the equation

$$\varphi(a) = Ge^{au}$$

then expresses that the function $\varphi(a)$ is generated from e^{au} by generalization.

This is in fact an extension of Liouville's generalized derivatives; the latter defined the derivative of index μ of the function $\varphi(a)$ as given by the equation

$$\frac{d^\mu \varphi}{da^\mu} = A_\alpha e^{a\alpha}\alpha^\mu + A_\beta e^{a\beta}\beta^\mu + A_\gamma e^{a\gamma}\gamma^\mu + \cdots;$$

and Oltramare proposed to construct a more general calculus* by considering expressions of the form

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* See Laisant's introduction to Oltramare's lithographed essay on the calculus of generalization published previously.

$$A_\alpha e^{a\alpha}\psi(\alpha) + A_\beta e^{a\beta}\psi(\beta) + A^\gamma e^{a\gamma}\psi(\gamma) + \cdots ;$$

the functions ψ not containing a; this expression he represents by the symbol $Ge^{au}\psi(u)$, which is clearly the result of an operation effected on $\varphi(a)$ if Ge^{au} represents $\varphi(a)$. Thus, for example, $Ge^{au}u^\mu$ is the μth differential coefficient of the function $\varphi(a)$. Since $e^{au}\psi(u)$ is any function of a and u, $Ge^{au}\psi(u)$ is equivalent to $Gf(u)$, and this last is the result of a certain operation on $\varphi(a)$; this operation is the object of the calculus of generalization; the inverse operation is called degeneralization.

These notions and their usefulness will perhaps be made clearer by a few examples taken from the treatise.

1° Assuming the identity

$$e^{au} = 1 + \frac{au}{1} + \frac{a^2u^2}{1.2} + \cdots + \frac{a^xu^n}{n\,!} + \cdots,$$

let us define φ by means of the equation

$$Ge^{au} = \varphi(x + a) ;$$

from this equation we deduce the following

$$G1 = \varphi(x),\ Gu = \varphi'(x),\ Gu^2 = \varphi''(x),\ \cdots,\ Gu^n = \varphi^{(n)}(x),$$

hence we have

$$\varphi(x + a) = \varphi(x) + \varphi'(x)\frac{a}{1} + \varphi^n(x)\frac{a^2}{1.2} + \cdots$$
$$+ \varphi^{(n)}(x)\frac{a^n}{n!} + \cdots,$$

Taylor's series.

Conversely, putting $\varphi(x)$ equal to e^{xu} we obtain

$$\varphi(x + a) = e^{(x+a)u}, \quad \varphi(x) = ue^{xu},\ \cdots,\ \varphi^{(n)}(x) = u^ue^{xu},$$

which, put in Taylor's series, give the development of e^{au} in series.

2° Let it be proposed to determine the function $\Psi(x, y)$ which satisfies the equation of finite differences

$$\Psi(x, y) - a\Psi(x - 1, y + 1) = 0.$$

We admit that this relation has been obtained by the generalization (using this word always in the technical sense ascribed to it above) of an identity which it is easy to determine.

As every function of two variables can be expressed by the generalizing function Ge^{xu+yv} let us put

$$\Psi(x, y) = Ge^{xu+yv}.$$

This value substituted in the proposed equation gives

$$Ge^{xu+yv}(1 - ae^{-u+v}) = 0\,;$$

whence, suppressing the factor e^{xu+yv} and degeneralizing, we have

$$1 - ae^{-u+v} = 0\,;$$

from which relation we deduce

$$e^u = ae^v\,;$$

raising to the xth power and multiplying by e^{yv} we have

$$e^{xu+yv} = a^x e^{(x+y)v},$$

which generalized gives

$$\Psi(x,y) = a^x \varphi(x + y),$$

the integral of the proposed equation, φ being an arbitrary function.

3° As another example take the integration of the linear equation

$$\frac{dz}{dx} + a\frac{dz}{dy} = bz.$$

Writing $$z = Ge^{xu+yv},$$

which is permissible since every function of two variables can be put in this form, and substituting this value for z in the given equation, the latter becomes

$$Ge^{xu+yv}(u + av - b) = 0,$$

whence, degeneralizing this expression and suppressing the factor e^{xu+yv}, we have

$$u + av - b = 0\,;$$

and hence $$e^{u+av-b} = 1, \text{ or } e^u = e^{b-av}\,;$$

which raised to xth power and multiplied by e^{yv} gives

$$e^{xu+yv} = e^{bx+(y-ax)v},$$

then the value of z will be expressed by generalizing the two members; we shall have

$$z = Ge^{xu+yv} = e^{bx}Ge^{(y-ax)v} = e^{bx}\Psi(y - ax),$$

where Ψ is an arbitrary function.

4° As a last example consider the generalizing function of the numbers of Bernouilli :

$$\frac{ue^u}{e^u - 1} = e^{Bu} = 1 + B_1 u + B_2 \frac{u^2}{1.2} + B_3 \frac{u^4}{4\,!} + \cdots;$$

by replacing u by hu we deduce

$$hue^{hu} = e^{(h + Bh)u} - e^{Bhu},$$

then putting $Ge^{xu} = F(x)$ as defining equation of the function $F(x)$ we have, by generalizing both members of this identity,

$$F(x + h + Bh) - F(x + Bh) = hF'(x + h),$$

the fundamental formula of the theory of Bernouillian numbers.

By multiplying both numbers of the identity above by $e^{nu} - 1$ we have

$$\frac{ue^u(e^{nu} - 1)}{e^u - 1} = e^{(n+B)u} - e^{Bu} = u(e^u + e^{2u} + \cdots + e^{nu})\,;$$

generalizing both members and putting $Ge^{xu} = f(x)$, we find the summation formula of Maclaurin

$$f'(1) + f'(2) + \cdots + f'(n) = f(n + B) - f(B).$$

These examples show the part to be played by the calculus of generalization when the operations are linear ; on the other hand, when the operations are non-linear the calculus is no longer applicable ; it may then be regarded as a theory of linear operations.

Its general problem as regards its own mechanism is to determine the value of the expression $G\Psi(u, v, w, \ldots)$ represented by the development of

$$\Psi(D_x, D_y, D_z, \ldots)\ \varphi(x, y, z, \ldots),$$

the function φ being defined by the relation

$$\varphi(x + a, y + b, z + c, \ldots) = Ge^{au + bv + cw + \cdots}$$

The order followed in the construction of the various forms $G\Psi$ and the applications to be made of these tools appear from the chapters of Oltramare's work which are in order 1° calculus of generalization, 2° generalization (technical use of word always) of functions of a single variable, 3° generalization of functions of several variables, 4° of rational functions, 5° of exponential functions, 6° of logarithmic functions, 7° of circular functions, 8° of various forms of transcendental functions, 9° expression of the integral ${}^n\!\!\int\varphi(x)dx^n$ as a definite integral, 10° differentiation and integration with fractional indices, 11° transformations of series into definite integrals and reciprocally, 12° expression of the sums of certain general series as definite integrals, 13° integration of equations, 14° determination of a particular integral of every linear differential, difference, or partial differential equation, having constant coefficients and second member, 15° inverse calculus of definite integrals, 16° integration of linear differential or difference equations with constant coefficients, 17° integration of linear partial differential or difference equations with constant coefficients, 18° integration of certain partial differential equations with variable coefficients, 19° integration of equations of mixed differences, 20° integration of simultaneous equations, 21° integration of certain equations which can be transformed into linear equations.

One of the principal advantages which the author claims for the method is the application which can be made of it to the integration of differential equations. Its rôle in higher analysis he likens to that of logarithms in numerical reckoning. It makes the integration of equations, difference and partial, single or simultaneous, depend on the solution of algebraic equations ; moreover, it permits of determining in general the maximum to the number of arbitrary functions entering an integral in order that it be complete, and in particular cases the process can determine the exact number of these arbitrary functions.

The general process employed by this calculus for the integration of equations consists in representing the known and unknown functions under the generalized forms, substituting these in the equations, and deducing the integrals by the aid of generalization.

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SHORTER NOTICES.

Jacob Steiner's Vorlesungen über synthetische Geometrie. Zweiter Theil: Die Theorie der Kegelschnitte gestützt auf projective Eigenschaften. Auf Grund von Universitätsvorträgen und mit Benutzung hinterlassener Manuscripte Jacob Steiner's. Bearbeitet von HEINRICH SCHRÖTER. Dritte Auflage, durchgesehen von RUDOLF STURM. Teubner, Leipzig, 1898. xvii + 537 pp. Price, 14 marks.

THE fact that this new edition of Steiner's lectures, edited and published by Schröter in 1866, has been prepared by