SOME THEOREMS CONCERNING LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

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I WISH to communicate to the Society certain results at which I have arrived, reserving the proofs and also further developments for another occasion. We shall be concerned with differential equations whose coefficients are not assumed to be analytic, and we consider the case in which $x = 0$ is a *regular singular point** of the equation, that is, the case in which the equation is of the form:

$$(1) \qquad \frac{d^2y}{dx^2} + \left(\frac{\mu}{x} + p_1\right)\frac{dy}{dx} + \left(\frac{\nu}{x^2} + q_1\right)y = 0,$$

where μ and ν are constants, and $|p_1|$ and $x\,|q_1|$ can be integrated up to the point $x = 0$.† For the sake of simplicity I shall not here refer to the case in which the exponents of the point $x = 0$ are equal, although this case might easily be included. By methods analogous to those which I have sketched in a simpler case (BULLETIN, Second Series, Volume 6, p. 100) we obtain the

FIRST THEOREM OF COMPARISON: *If the exponents of $x = 0$ in* (1) *are real and unequal, and y_1 is the solution of* (1) *which corresponds to the larger exponent, and if y_1 vanishes when $x = x_1 > 0$, but does not vanish between* 0 *and x_1; if moreover we have a second equation of the same form as* (1) *which differs from it only in that throughout the interval $0 < x \leqq x_1$ the coefficient of y in the second equation $\geqq \nu/x^2 + q_1$ (the equality sign not holding throughout the whole interval); and finally if y_2 is the solution of this second equation which corresponds to the larger exponent of $x = 0$; then y_2 has at least one root in the interval $0 < x < x_1$.*

In order to compare the solutions corresponding to the smaller exponents, we must restrict our differential equation further by assuming that a constant $k < 1$ can be found such that $x^k p_1$ and $x^{k+1} q_1$ remain finite as x approaches zero. We will refer to this as *restriction* (A). We then have the

THEOREM: *The solution of* (1) *corresponding to the exponent*

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* For the meaning of this and other terms used, see my paper in the January number of the *Transactions*, p. 40; or an earlier paper in the BULLETIN, 2d Series, vol. 5, p. 275.

† For a more precise statement see my papers just referred to.

with the smaller real part can be developed by the method of successive approximations exactly as the solution corresponding to the exponent with the larger real part is developed in my article in the Transactions, provided that $R\varkappa < 1 - k$, *where* $\varkappa$ *is the difference of the exponents at* $x = 0$ *so taken that* $R\varkappa > 0$.

By using this theorem we deduce the

SECOND THEOREM OF COMPARISON: *If the conditions of the first theorem of comparison are fulfilled, and if moreover both equations have the same exponents, and both satisfy restriction* (A); *and if* $\bar{y}_1$ *and* $\bar{y}_2$ *are the solutions corresponding to the smaller exponent; and if* $\bar{y}_1$ *vanishes when* $x = \bar{x}_1 > 0$, *but not in the interval* $0 < x < \bar{x}_1$, *then* $\bar{y}_2$ *will vanish at least once in this interval provided that* $\varkappa < 1 - k$.

Finally I should like to mention a fact which had escaped my notice until after my paper in the *Transactions* was printed, namely that the class of singular points which I there discuss under the name *regular* can be brought into very close connection with the class of singular points previously studied by Kneser (*Crelle's Journal*, Volumes 116, 117, 120; *Mathematische Annalen*, Volume 49). This can be done by replacing the independent variable x, which I use, by z where $x = e^{-z}$. Although many of my results can be deduced by this method from those previously found by Kneser and *vice versa*, the results in the two cases are by no means coextensive, nor does either include the other. I shall come back to this matter more at length on a subsequent occasion. It may be noted that the method of successive approximations can also be applied directly to Kneser's case.

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NOTE ON THE ENUMERATION OF THE ROOTS OF THE HYPERGEOMETRIC SERIES BETWEEN ZERO AND ONE.

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IN the May number of the BULLETIN for 1897, the writer gave a solution of the problem of enumerating the real roots of $F(\alpha, \beta, \gamma, x)$ between zero and one which depended on two well known theorems of Sturm—there referred to as [A]