It follows that $I''$ is identical with $SH(6, 2^n)$. But $I''$ is holoedrically isomorphic with $G_{p=3}$ and therefore with $HA(4, 2^n)$, whose second compound is $G_{p=2}$.

4. **Note.**—It appears that the quaternary transformation group which naturally corresponds to the finite group $\text{HO}(4, 2^n)$ is not continuous.

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THE HESSIAN OF THE CUBIC SURFACE. II.

BY DR. J. I. HUTCHINSON.

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The aim of the following paper is to extend the results of a previous article on the same subject (Bulletin, March, 1899, p. 282) by determining all the quintic and sextic curves on the Hessian of the cubic surface, and giving some theorems connected with them, and with the quartic curves already determined.

I will write the equation of the Hessian in the form

$$F = xyzu + wyzu + wxzu + wxyu + wxyz = 0,$$

where $v, x, y, z, u$ are connected by the relation

$$aw + bx + cy + dz + eu = 0,$$

in which $a, b, c, d, e$ are arbitrary constants.

As already shown, the surface $F$ contains three classes of biquadratic curves, viz.:

$a$. A class containing 15 families which lie on 30 families of cones, all the cones of the same family cutting $F$ in two lines and tangent along a third.

$a'$. A class containing 30 families of curves lying on 30 families of cones tangent to $F$ along two lines.

$a''$. A class containing 15 families of curves determined by as many families of quadrics each intersecting $F$ in a gauche quadrilateral, and by 30 families of quadrics each meeting $F$ in two lines and a conic.

Consider the family of $a_1$ determined by the cones

$$A_1 = x(w + y) + \lambda wy = 0,$$

$$A_1' = x(z + u) + (1 - \lambda)zu = 0.$$
The plane \( w + y = 0 \) is tangent to all of the cones \( A_i \) and hence is a double tangent plane for each biquadratic curve of the family. The two points of tangency with each curve form pairs of points in an involution on the line \( wy \). Similarly, the plane \( z + u = 0 \) is a double tangent plane for each curve of the family, the points of contact forming an involution on the line \( zu \).

All the curves of the family pass through the four nodes of \( F \) which lie on the lines \( wx \) and \( xy \) excluding their point of intersection (or we may say, the four which lie in the plane \( x \) excluding the vertices of \( A_i \) and \( A'_i \)). These are the only points common to two curves of the same family.

The 15 families of \( a_i \) can be grouped in 5 sets of 3 each, any one set having the property that a cubic surface can be passed through three curves arbitrarily chosen, one from each family of the set. For example, with the family determined by equations (1) are associated the two others obtained by the permutations \( (wz) \) and \( (wu) \). The family of cubic surfaces which intersects \( F \) in these three families of curves has the equation

\[
A_i [z + \lambda'(1 - \lambda'')z + \lambda''(1 - \lambda')u] + A'_i [(1 - \lambda')(1 - \lambda'')w + z + \lambda''y] = 0,
\]

since the left member is unaltered by the permutations \( (wz)(\lambda\lambda') \) and \( (wu)(\lambda\lambda'') \).

Consider the family \( a_s \) determined by the intersection of the quadrics

\[
A_s \equiv wy + \lambda xu = 0,
\]

\[
A'_s \equiv wx + wy + xy - \lambda x(z + u) = 0.
\]

Each curve of the family passes through the same four nodes of \( F \) which form the points of intersection of the curve of the family (1) of \( a_i \). These two families of \( a_i \) and \( a_s \) are also associated in such a way that they form the complete intersection with \( F \) of the family of quadrics

\[
A_i - \lambda'A'_i \equiv (\lambda - 1)A_s + A'_s = 0.
\]

The four points in which a given curve of the family (2) meets the lines \( wx, wu, yz, yu \) lie in a plane whose equation is

\[
y + w - \lambda(z + u) = 0.
\]

The plane \( w + u = 0 \) intersects \( A_i \) in two lines one of which is the line \( wu \). The plane is accordingly tangent to
$A_3$ in the point whose coordinates are $w = u = 0$, $y = \lambda x$. These coordinates also satisfy the equation $A_3' = 0$. Hence, the plane $w + u$ touches each biquadratic of the family in that point of the line $wu$ where it is tangent to the quadric $A_3$ containing the given curve. Similarly for the planes $w + z$, $y + z$, $y + u$.

By means of the permutations $(wz)$ and $(wu)$ applied to the equations (2) we determine two other families which together with the first are so related that a cubic surface can be passed through three curves arbitrarily chosen, one from each family. The equation of the cubic is

$$A_3[\lambda''w + (1 + \lambda')(1 + \lambda'')x + \lambda'y] - A_3'[\lambda''w + \lambda'y - \lambda'\lambda''z - u] = 0,$$

since the left member is unaltered by the permutations $(yw)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

The curves (1) of $a_2$, and (2) of $a_3$ have in common the property that they meet any of the conics \{xw\} (in the plane $w + x = 0$), \{xy\}, \{xz\}, \{xu\} in pairs of points forming an involution.

Any curve of (1) intersects any curve of (2) in 8 points, on account of the identity

$$A_1 - \mu A_1' + (1 - \lambda)A_3 - A_3' \equiv 0,$$

where $A_1$, $A_1'$ contain the parameter $\lambda$, and $A_3$, $A_3'$ contain $\mu$. But since

$$(1 - \lambda)A_1 + \lambda\mu A_1' = \lambda(1 - \lambda)A_3 \equiv x[(1 - \lambda)(w + y) + \lambda\mu(z + u)],$$

it follows that the four movable points of intersection lie in the plane

$$(1 - \lambda)(w + y) + \lambda\mu(z + u) = 0,$$

the remaining four being in the plane $x = 0$ as already noticed.

Consider the family of $a_3$ determined by the cones

$$(3) A_3 \equiv \lambda w^2 + wx + wy + xy = 0,$$

$$A_3' \equiv \lambda(zu + wz + wu) + (x + y)(z + u) = 0.$$

Two curves of the same family do not intersect. The planes $w + x$, $w + y$, and $z + u$ are double tangent planes to each curve of the family, the pairs of points so de-
termined forming involutions on the lines \(wx, wy,\) and \(zu\) respectively.

The four lines \(xz, yz, xu, yu\) are met by each curve in one point each, the four points so determined lying in the plane

\[
\lambda w + x + y = 0.
\]

If we apply to equations (3) the permutations \((wxz)\) and \((wzx)\), we obtain two other families of curves which together with (3) form the complete intersections of \(F\) and a family of cubic surfaces whose equation is

\[
A_4[\lambda'x + y + \lambda''(1 - \lambda')z - (1 - \lambda')(1 - \lambda'')u] + A_4'[(\lambda'\lambda'' - \lambda'' + 1)w + \lambda'\lambda''x + y + \lambda''z] = 0.
\]

The function in the left member is unchanged by the permutation \((wxz)(\lambda'\lambda'')\). By means of the permutations \((xy), (zu),\) and \((xy)(zu)\) which leave \(A_4\) and \(A_4'\) unchanged we see that the family (3) is also associated in a similar manner with three other pairs of families.

There are four classes of quintic curves, all of the first species according to Salmon's classification. (Geometry of three dimensions, 4th edition, p. 318.)

There are four classes of quintic curves, all of the first species according to Salmon's classification. (Geometry of three dimensions, 4th edition, p. 318.)

\[B_1 \equiv \mu_1xy + \mu_2wy + \mu_3wx = 0.\]

The same family of quintics is also determined by the partial intersection of \(F\) and the three families of cubic surfaces

\[
\begin{align*}
\Gamma_1 &\equiv \mu_1xy(z + u) + zu[(\mu_1 - \mu_2)x + (\mu_1 - \mu_2)y] = 0, \\
\Gamma_2 &\equiv \mu_2wy(z + u) + zu[(\mu_2 - \mu_3)w + (\mu_2 - \mu_1)y] = 0, \\
\Gamma_3 &\equiv \mu_3wx(z + u) + zu[(\mu_3 - \mu_4)w + (\mu_3 - \mu_1)x] = 0,
\end{align*}
\]

since

\[
\begin{align*}
\mu_1F &\equiv w\Gamma_1 + zuB_1, \quad \mu_2F \equiv x\Gamma_2 + zuB_1, \quad \mu_3F \equiv y\Gamma_3 + zuB_1,
\end{align*}
\]

If \(\Gamma'\) denote the function \(\Gamma'\) with \(\mu_2'\) written in the place of \(\mu_2\), it is easy to see that the two curves whose parameters are \(\mu_2\) and \(\mu_2'\) lie on the cubic surface \(\Gamma\), where

\[
\Gamma \equiv \mu_1\Gamma_1 + \mu_2'\Gamma_2 + \mu_3'\Gamma_3 \equiv \mu_1\Gamma_1 + \mu_2\Gamma_2 + \mu_3\Gamma_3.
\]

For \(\mu_2' = \mu_2\) we obtain a family of cubic surfaces inscribed to \(F\) along the quintics of the family (4) and the line \(zu\).
By taking $\mu_i = 0$ we see that this family of quintics includes three families of plane cubics, and by taking $\mu_i = \mu_k$ we find that it also includes three families of biquadratics of $a_i$.

Every curve of the family passes through the seven nodes lying on the lines $wx$, $wy$, and $xy$. Any two curves intersect in two other points whose joining line is evidently a common generator of the two cones on which the two curves lie.

Through any two curves of $\beta_i$, whether belonging to the same family or not, a cubic surface can be passed which meets $F$ elsewhere in two lines.

$\beta_i$. The next class of quintics contains sixty families determined by sixty families of cones, each passing through intersecting lines of $F$ and tangent along one of them. The equation

$$B_2 \equiv ay^2 + \beta wy + \gamma x(w + y) = 0$$

leads to a family of this class which is also determined by the cubic surfaces

$$G \equiv ay(xs + xu + xu) + (a - \beta + \gamma)xu + \gamma u(z + u) = 0,$$

since

$$(a - \beta)F \equiv G(w + y) - B_2(xs + uz + xu).$$

Every curve of the family passes through the three nodes which lie on the line $xy$. Any two of the curves intersect in two other points.

$\beta_i$. The third class contains sixty families determined by sixty families of quadrics passing through two non-intersecting lines of $F$ and a third line cutting both. The two equations

$$B_3 \equiv y(\lambda w + \mu x) + wz = 0,$$
$$H_3 \equiv xu(w + z) - (\lambda w + \mu x)(yu + xu + xy) = 0,$$

determine such a family, since

$$F \equiv yH + B_3(yu + xu + xy).$$

Every curve of the family passes through the five nodes situated on the lines $wz$, $wy$, and $yz$ excluding the two points of intersection. Any two curves have two other points of intersection.

If we apply to equations (6) the permutations $(wx)(zu)$, $(yz)(zu)$, $(uw)(yu)$, $(wu)(xy)$, we obtain together with (6) five families of $\beta_i$ having the property that if two curves be
taken from any two different families of the set, they will lie on a cubic surface which either passes through a conic, or is tangent to $F$ along a line. The 60 families of $\beta_1$ group themselves into 12 sets of 5 of the like character.

$\beta_1$. The fourth class contains 30 families determined by 30 families of quadrics meeting $F$ in a line and a conic. Such a family is given by the equations

$$
B_i \equiv zu + yu + yz - (w + x)(\lambda y + \mu z) = 0,
$$

$$
J \equiv \lambda wxy + \mu wxz + yzu = 0,
$$

since

$$
F \equiv wxB_i + (w + x)J.
$$

Any two curves of the family intersect in five points, three of which are the nodes contained on the line $yz$.

There are six different classes of sextic curves on the Hessian.

$\gamma_1$. The first class contains 30 families, each of which is determined by the quadrics passing through two intersecting lines of $F$. The curves determined by the equation

$$
C_i \equiv wp + xy = 0,
$$

where

$$
p = aw + \beta x + \gamma y + \delta z + \epsilon u,
$$

also lie on the cubic surfaces

$$
K_i \equiv -zwp + xyz + xyu + xu + yzu = 0,
$$

since

$$
F \equiv zuC_i + wK_i.
$$

Any two curves of this family with parameters $a, \ldots$ and $a_1, \ldots$ intersect in the four nodes lying on the lines $wx$ and $wy$, excepting their common point, and in six other points lying on a conic in the plane $p - p_i = 0$.

A second family generated by the equations

$$
\bar{C}_i \equiv wp' + zu = 0,
$$

$$
\bar{K}_i \equiv -xyp' + yxz + yzu + xu + yzu = 0
$$

is associated with the preceding in such a way that a curve from each family lies on the cubic surface

$$
K \equiv p'C_i - K_i \equiv p\bar{C}_i - \bar{K}_i = 0.
$$

$\gamma_2$. The second class, containing 10 families, is determined by quadrics which touch $F$ along a line. Such a family is

$$
C_i \equiv (w + x)p + uxx = 0,
$$

also determined by the surfaces

$$
K_2 \equiv -yue + p(yz + yu + zu).
$$
In this case we have
\[ F \equiv C'_1(yz + yu + zu) - (w + x)K_s. \]

\[ \gamma_s. \] The third class also contains 10 families. The quadric
\[ C_s \equiv (w + x)p + yz + yu + zu = 0, \]
which meets \( F \) in a conic, and the cubic surface
\[ K_s \equiv uxp - yzu = 0 \]
determine such a family, since
\[ F \equiv wxC_s - (w + x)K_s. \]

Any two curves of the same family in either \( \gamma_s \) or \( \gamma_s' \) intersect in six points lying on a conic.
A curve from each of the two families of \( \gamma_s \) and \( \gamma_s' \) just mentioned lies on the cubic surface
\[ K \equiv C_p' + K_s \equiv C_p'p + K_s. \]

\[ \gamma_s'. \] The fourth class contains 15 families, each determined by the intersection with \( F \) of a family of quadrics such as
\[ awy + bwy + cxy + dxx = 0, \]
passing through two non-intersecting lines \( wx, yz \). A sextic of this class cannot lie on a cubic surface.

\[ \gamma_s'. \] The fifth class of sextics contains 60 families. These curves are formed by the partial intersection of two cubic surfaces such as
\[ \Delta_1 \equiv y(wx + wz + xz) + z(axw + bxx + cwx) = 0, \]
\[ \Delta'_2 \equiv u(axw + bxx + cwx) - wx(y + u), \]
which intersect in the three lines \( wx, wz, xz \), and a sextic on \( F \), since
\[ F \equiv u\Delta_1 - z\Delta'_2. \]

\[ \gamma_s'. \] The sixth class contains a single family, the Steiner sextics, determined by the equation
\[ S \equiv xxyz + lwy + muwx + rvwx = 0, \]
or equally well by four other equations referred to the four other fundamental tetrahedra.

Let \( s \) and \( s' \) be any two curves of the family and let \( T = 0, T' = 0 \) be the two cubic surfaces which touch \( F \)
along \( s \) and \( s' \) respectively. Also let \( H = 0 \) be the cubic
surface containing both $s$ and $s'$. Then an identity of the form

\[(1) \quad QF = TT' + H^2\]

necessarily exists, where $Q$ is a function of the second degree. A course of reasoning similar to that used by Humbert in connection with the Kummer surface enables us to draw a number of interesting conclusions from (1).

The two curves $s$ and $s'$ intersect in four points $p_1, p_2, p_3, p_4$ (besides the 10 nodes of $F$ through which every curve of the family passes). The five surfaces $Q, F, T, T', H$ are mutually tangent at each of these four points.

The surface $T$ intersects $H$ in $s$ and a twisted cubic $c$. It is evident that $c$ lies on $Q$ and that $T$ touches $Q$ along this cubic. Similarly, $T'$ touches $Q$ along a cubic $c'$ which also lies on $H$.

The 24 points of intersection of $Q, H,$ and $F$ are double points of the surface $TT' = 0$. Among these occur the points $p_i$, each counted four times. The remaining eight points, since they lie on the curves $s$ and $s'$, are not points of tangency of $T$ and $T'$. They must therefore be nodes on either $T$ or $T'$. It is evident that half of them are on the one, and half on the other surface. Let $a_1, a_2, a_3, a_4$ be the four nodes on $T$. The joining line of any two of these points lies entirely on $T$, and is accordingly tangent to $F$ in a third point. Hence, the tetrahedron whose vertices are the nodes of the cubic surface $T$ which touches $F$ along $s$ is inscribed to $F$ as to its vertices and circumscribed to $F$ as to its edges. There are evidently $\simeq 8$ such tetrahedra.

The four points $a_i$ and the points $p_i$ counted twice, form the complete intersection of the cubic $c$ with $F$.

The cubic surfaces $S$ and $S'$ defining the sextics $s$ and $s'$ intersect in six lines and a twisted cubic $k$. The latter passes through the four points $p_i$ and the four vertices of the tetrahedron of reference. Conversely, every twisted cubic $k$ passing through the vertices of the tetrahedron of reference intersects $F$ in four remaining points $p_i$, which form the four points of intersection of two sextics $s$ and $s'$ (and hence are the common points of a singly infinite system of sextics.)

The Hessian $F$ is invariant for the birational transformation $T_z$ defined by the equations

\[
aw' : bx' : cy' : dz' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.
\]

This transformation interchanges the family of cubics $k$ with the lines of space, and the sextics $s$ with the plane.
quartics of $F$. Four other transformations of a similar character may be obtained by starting from any of the other four tetrahedra contained in the fundamental pentahedron. These are

$$T_1 : \frac{bx'}{x} : \frac{cy'}{y} : \frac{dz'}{z} : \frac{ew'}{u}$$

$$T_2 : \frac{aw'}{w} : \frac{cy'}{y} : \frac{dz'}{z} : \frac{eu'}{u}$$

$$T_3 : \frac{aw'}{w} : \frac{bx'}{x} : \frac{dz'}{z} : \frac{eu'}{u}$$

$$T_4 : \frac{aw'}{w} : \frac{bx'}{x} : \frac{cy'}{y} : \frac{eu'}{u}$$

These five operations, each of period 2, generate a group $G$ of infinite order, since the operation $T_j T_i$ is of infinite period. As far as the points of $F$ are concerned, each transformation $T_i$ has exactly the same effect. In other words, there exists a subgroup of index 2 under the group $G$ for which each point of $F$ is unchanged in position.

Since a plane quartic has 28 bitangents it follows that the sextic $s$ has 28 bitangent cubics $k$ out of each of the five families of such cubics. Suppose the plane of the quartic $q$ corresponding to $s$ to be rotated about one of the bitangents of the curve. The intersections of the moving plane with $F$ will form a single infinity of plane quartics having the same line for bitangent. Hence, for a given sextic $s$ there are 28 subfamilies of $\infty^1$ sextics, each family being tangent to $s$ in a pair of points, or in other words, the four points $p_i$ fall together in pairs 28 times on a given sextic.

Two sextics do not in general touch each other at a node of $F$. There are $\infty^2$ sextics which touch each other and have a given generator of the tangent cone at the node for a common tangent line.

Suppose that we consider the $\infty^1$ of these curves which are also mutually tangent at a second node of $F$. If the joining line of the first and second nodes is a line of $F$, then these curves are also mutually tangent in two other nodes of $F$, the four nodes together forming the vertices of one of the five fundamental tetrahedra. If the first and second nodes do not have a line of $F$ for joining line then the two remaining points of intersection determine a line which meets two of the lines of $F$. For example, if the two nodes be

NOTE ON THE GROUP OF ISOMORPHISMS.

BY DR. G. A. MILLER.

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Let \( s_1, s_2, \ldots, s_g \) represent all the operators of a group \( G \) and let \( t_a s_a \) correspond to \( s_a \) (\( a = 1, 2, \ldots, g \)) in any given simple isomorphism of \( G \) with itself. It is evident that \( t_a \) is some operator of \( G \). When \( G \) is abelian these \( t_a \)'s must constitute a group \( T \) which is isomorphic with \( G \).* In this isomorphism, \( t_a \) evidently can not be the inverse of \( s_a \) unless \( s_a = 1 \). As this condition is sufficient as well as necessary, we have

**Theorem I.**—Every simple isomorphism of an abelian group \( A \) with itself may be obtained by 1° making \( A \) isomorphic with one of its subgroups or with itself in such a manner that no operator corresponds to its inverse, and 2° making each operator of \( A \) correspond to itself multiplied by the operator which corresponds to it in the given isomorphism.

The simplest case that can present itself is the one in which the subgroup of \( G \), which corresponds to identity of \( T \) in the given isomorphism between \( G \) and \( T \), includes \( T \). The resulting simple isomorphism of \( G \) with itself must correspond to an operator in the group of isomorphisms of \( G \), whose order is equal to the operator of highest order in \( T \). When the order of \( T \) is an odd prime number \( p \), or the double of an odd prime, only one other case can present itself; viz, the case in which \( T \) corresponds to itself, or to its subgroup of an odd prime order, in the given isomorphism between \( G \) and \( T \). The resulting simple isomorphism of \( G \) with itself may clearly correspond to a cyclical group of order \( p - 1 \), or to any one of its subgroups in the group of isomorphisms of \( G \). These results lead to the following

* When \( G \) is non-abelian, these \( t_a \)'s need not constitute a group, as can be seen from the simple isomorphisms of the symmetric group of order 6 with itself.