ON GROUPS OF ORDER 8!/2.

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§ 1. In the BULLETIN, vol. IV. (1898), pp. 495–510, Dr. L. E. Dickson discusses the structure of the hypoabelian groups. Among the simple groups of the system J, occurs one of order $8!/2$ ($p^n = 2^1$, $m = 3$); this 2.3 or senary linear group is defined as the totality of linear substitutions on 2.3 indices, as follows:

$$(1)\qquad \xi_i' = \sum_{j=1}^{3} (\alpha_j^{(i)} \xi_j + \gamma_j^{(i)} \eta_j), \quad \eta_i' = \sum_{j=1}^{3} (\beta_j^{(i)} \xi_j + \delta_j^{(i)} \eta_j), \qquad (i = 1, 2, 3),$$

satisfying the relations

$$(2)\qquad \sum_{i=1}^{3} \begin{vmatrix} \alpha_j^{(i)} & \gamma_j^{(i)} \\ \beta_j^{(i)} & \delta_j^{(i)} \end{vmatrix} = 1, \qquad \sum_{i=1}^{3} \begin{vmatrix} \alpha_j^{(i)} & \gamma_k^{(i)} \\ \beta_j^{(i)} & \delta_k^{(i)} \end{vmatrix} = 0,$$

$$\sum_{i=1}^{3} \begin{vmatrix} \alpha_j^{(i)} & \alpha_k^{(i)} \\ \beta_j^{(i)} & \beta_k^{(i)} \end{vmatrix} = 0, \qquad \sum_{i=1}^{3} \begin{vmatrix} \delta_j^{(i)} & \delta_k^{(i)} \\ \delta_j^{(i)} & \delta_k^{(i)} \end{vmatrix} = 0,$$

$$(j \neq k;\ j, k = 1, 2, 3);$$

$$(3)\qquad \sum_{j=1}^{3} \beta_j^{(i)} \delta_j^{(i)} = 0, \quad \sum_{j=1}^{3} \alpha_j^{(i)} \gamma_j^{(i)} = 0, \quad \sum_{i,j=1}^{3} \alpha_j^{(i)} \delta_j^{(i)} = m,$$

$$(i = 1, 2, 3;\ m = 1, 2, 3).$$

*The present paper determines that the above group is abstractly the alternating group $G^8_{8!/2}$, and thus establishes a new proof of its simplicity.**

Writing the substitutions (1) in square array, and considering the elements of the group as the matrices of these coefficients, we have

*Dr. L. E. Dickson in the *Proc. of the Lond. Math. Soc.*, vol. 30, "The structure of certain linear groups with quadratic invariants," pp. 81 et seq., has proved the correspondence of these groups.

$$\begin{pmatrix} \alpha_1^{(1)} & \gamma_1^{(1)} & \alpha_2^{(1)} & \gamma_2^{(1)} & \alpha_3^{(1)} & \gamma_3^{(1)} \\ \beta_1^{(1)} & \delta_1^{(1)} & \beta_2^{(1)} & \delta_2^{(1)} & \beta_3^{(1)} & \delta_3^{(1)} \\ \alpha_1^{(2)} & \gamma_1^{(2)} & \alpha_2^{(2)} & \gamma_2^{(2)} & \alpha_3^{(2)} & \gamma_3^{(2)} \\ \beta_1^{(2)} & \delta_1^{(2)} & \beta_2^{(2)} & \delta_2^{(2)} & \beta_3^{(2)} & \delta_3^{(2)} \\ \alpha_1^{(3)} & \gamma_1^{(3)} & \alpha_2^{(3)} & \gamma_2^{(3)} & \alpha_3^{(3)} & \gamma_3^{(3)} \\ \beta_1^{(3)} & \delta_1^{(3)} & \beta_2^{(3)} & \delta_2^{(3)} & \beta_3^{(3)} & \delta_3^{(3)} \end{pmatrix} \quad \text{with the identity matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\gamma_j^{(i)}$, $\delta_j^{(i)}$ are the marks 0 and 1 of the $G.F.[2^1]$.

These matrices compound according to the general law of composition of matrices

$$A = (a_{ij}), \quad B = (b_{jk}), \quad C = (c_{ik}), \quad AB = C, \quad \sum_{j=1}^{6} a_{ij} b_{jk} = c_{ik}.$$

§ 2. Access to the above described group is obtained through the Dickson generators $M_i M_j$, $N_{ij\lambda}$ ($\lambda = 1$; $i, j = 1, 2, 3$),* of the table p. 442, and elements E_1', E_2', $\cdots$, E_6' are determined having the following properties:

$$\sum (E_1', \cdots, E_6') \; \{E_1'^3 = E_{i+1}'^2 = (E_i' E_{i+1}')^3 = (E_i' E_j')^2 = I\}$$
$$(i,\ i+1, j = 1,\ 2, \cdots,\ 6\ ;\ i+1 < j),$$

where I is the identity element.

§ 3. Making use of Theorems B and C of Professor Moore's paper "Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holoedrically isomorphic with the symmetric and alternating groups on k letters," *Proceedings of the London Mathematical Society*, Dec. 10, 1896, vol. 28, pp. 358–359, the definite conclusions are arrived at that the generators $E_1', \cdots, E_6'$ satisfying the relations $\sum(E_1', \cdots, E_6')$, with order greater than the limit prescribed in the above theorem C, and therefore, by theorem C, satisfying no further relations, generate a group $G(E_1', \cdots, E_6')$, and this group $G(E_1', \cdots, E_6')$ is holoedrically isomorphic to the alternating group of degree eight $G^8_{8!/2}$, and is therefore simple; but the original senary group, as Dr. Dickson † has proved, is of order 8!/2 ; it is therefore identical with its, as we may say, alternating subgroup $G(E_1', \cdots, E_6')$, *and is therefore simple.*

§ 4. The elements $M_i M_j$, P_{ij} ‡ ($i, j = 1, 2, 3$), taken in pairs, thus $M_1 M_2$, P_{12}; $M_1 M_3$, P_{13}; $M_2 M_3$, P_{23}, play a most im-

* BULLETIN, loc. cit., p. 496, § 8; $N_{ij\lambda}(\lambda = 0)$ is the identity element in the $G.F.[2^n]$.

† loc. cit., § 5.

‡ BULLETIN, loc. cit., § 3 ; $M_i M_k$ transformed by $N_{ij1} = Q_{ij1}$, Q_{ij1} transformed by $Q_{ji1} = P_{ij}$.

portant rôle ; the discovery of their simple combinational properties forms the basis of this paper, namely the determination of S, an element of period seven,

$$S = (M_1M_2 \,.\, N_{121} \,.\, N_{131} \,.\, P_{12} \,.\, P_{23})\,;$$

the element T of period three

$$T = (N_{121} \,.\, M_1M_2)^2\,;$$

and the elements W and R defined in the next paragraph.

§5. The elements E_1', $\cdots$, E_6' were derived from suitable combinations of the two elements, W of period four and R of period fifteen,

$$W = (TS^4 . P_{13}), \quad R = (TS^4 . M_1M_3)\,;$$

in fact E_1', $\cdots$, E_6' are identically the functions of W and R that $e_1, \cdots, e_6$ are of c_0 and c_1 in Prof. Moore's article : "Concerning the general equations of the seventh and eighth degrees," *Mathematische Annalen*, vol. 51, p. 437, in which the identity is established of the alternating group of degree eight $G^8_{8!/2}$, and a certain quaternary group. R and W enjoy the following properties : *

$$\Sigma(R,\ W)\,;\ \{\, W^4 = R^{15} = (WR^5)^6 = (WR^{10})^4 = (WR^9)^6 = (WR^3)^4 = (WR^6)^4 = \mathrm{I} \,\}.$$

§ 6. *Table exhibiting the matrices corresponding to the notation* M_iM_j, N_{ij}, etc.†

$$M_1M_2 = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \quad M_1M_3 = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{pmatrix} \quad N_{121} = \begin{pmatrix} 1&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \quad N_{131} = \begin{pmatrix} 1&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

$$P_{12} = \begin{pmatrix} 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \quad P_{13} = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix} \quad P_{23} = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$$

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* The relations $\Sigma(R,\ W)$ are the same as the relations $\Sigma(C_0,\ C_1)$ of Prof. Moore's "Congruence group," loc. cit., p. 436, marked (*).

† The Dickson generators $M_2M_3 = M_1M_2 \cdot M_1M_3$ and N_{231} are not used in the present paper.

$$S = \begin{pmatrix} 1&0&0&1&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&1&1&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix} \quad T = \begin{pmatrix} 1&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad R = \begin{pmatrix} 0&1&1&1&1&1\\ 1&0&0&1&0&1\\ 0&1&0&1&1&0\\ 0&0&0&1&0&1\\ 1&0&0&0&0&1\\ 0&0&1&1&1&1 \end{pmatrix} \quad W = \begin{pmatrix} 0&0&1&1&1&1\\ 1&0&0&0&0&1\\ 0&1&0&1&1&0\\ 0&0&0&1&0&1\\ 1&0&0&1&0&1\\ 0&1&1&1&1&1 \end{pmatrix}$$

$$E_1' = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&1&1&0&1\\ 1&0&0&1&0&1\\ 1&0&1&0&0&1\\ 0&0&0&0&0&1\\ 1&0&1&1&1&1 \end{pmatrix} \quad E_2' = \begin{pmatrix} 0&1&0&1&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&1&0&0&1\\ 0&1&1&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \quad E_3' = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&1&1&0&1\\ 1&0&0&0&0&0\\ 1&0&1&0&0&1\\ 1&0&1&0&1&0\\ 0&0&0&1&0&0 \end{pmatrix} \quad E_4' = \begin{pmatrix} 0&1&1&1&1&1\\ 0&0&0&0&1&0\\ 0&1&1&0&1&0\\ 0&1&0&1&1&0\\ 0&1&0&0&0&0\\ 1&1&1&1&1&0 \end{pmatrix}$$

$$E_5' = \begin{pmatrix} 0&1&0&1&0&0\\ 0&0&1&0&0&1\\ 0&1&0&0&0&1\\ 1&0&1&0&0&1\\ 1&0&1&1&1&1\\ 0&0&0&0&0&1 \end{pmatrix} \quad E_6' = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&1&0&1&0\\ 0&0&0&0&1&0\\ 1&0&1&0&0&1\\ 0&0&1&0&0&0\\ 1&0&0&1&1&0 \end{pmatrix}$$

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LOBACHEVSKY'S GEOMETRY.

(SECOND PAPER.)

LOBACHEVSKY'S earliest published work on geometry, translated by Engel under the title "Ueber die Anfangsgründe der Geometrie," * contains the elements of a system of analytic geometry under the hypothesis of a variable angle of parallelism, together with numerous applications to the determination of the lengths of arcs, areas, and volumes. Some of this matter appears also in Lobachevsky's article on "Géométrie imaginaire" (Crelle, vol. 17) and more in his "Pangéométrie" (Kasan, 1856); but it is probably safe to say that the knowledge of this part of his work is not so general as that of the more elementary side of his theory, partly because of the difficulties involved in reading the last mentioned articles, and partly because of the fact that the widely known "Geometrische Untersuchungen" does not

* See the BULLETIN for May, 1900, p. 339.