of circular transformations of the plane. These with their appropriate symbols and invariant figures are as follows:

1. mG_6. No invariant figure.
2. $mG_4(A)$. A single invariant point.
3. $mpG_3(A)$. A single invariant point.
4. $mG_3(\bar{C})$. A real invariant circle.
5. $mG_3(i\bar{C})$. An imaginary invariant circle.
6. $mpG_2(A)$. A single invariant point.
7. $mG_2(AA')$. An invariant point pair.
8. $mhG_2(A\bar{C})$. An invariant point and a circle through it.
9. $mhG_1(AA'\bar{C})$. A pair of invariant points and a pair of orthogonal circles.
10. $mpG_1(A\bar{C})$. An invariant point and an invariant circle.
11. $meG_1(AA'C)$. An invariant point pair and an invariant circle.

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PURE MATHEMATICS FOR ENGINEERING STUDENTS.

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I. Its Utility.

I have had opportunity to become acquainted with the written opinions of graduates of from one to many years' standing, with regard to the benefit and utility of their instruction in all departments of technical work, and have also taken the opportunity of conferring personally with graduates in respect to my own department. In discussing this subject, I shall therefore appear, not as a special pleader for pure mathematics, but as one who proposes to present its claims in their true proportions to the other necessary work of the student.

In the first place, mathematical analysis is not so directly useful to the average engineer as the mathematician might expect. The mathematics that an engineer is obliged to use regularly is of that cut and dried form which is found tabulated in engineering handbooks of easy access, so that only a little arithmetic or algebra, and occasionally some trigonometry, answer all practical purposes. Seventy-five per cent. of graduates of technical institutes, now in actual practice, have probably forgotten how to differentiate and integrate, although most of them could easily refresh their memory in this respect if it were necessary. We can say, in favor of the calculus, that the remaining twenty-five per cent. include those who are leaders in their profession, and those who occupy positions under which new conditions are constantly arising and in which originality of thought is required ; and among these are many who have found use for all the mathematics that they could acquire.

The question arises, when so few engineering students afterwards make practical use of the higher mathematics, Is it advisable to teach it thoroughly to all students? The answer that I find to this question is, yes. Almost every practical engineer who has been properly taught, sees the benefit of such studies. Such engineers will generally grant of their own accord the practical benefits of their mathematical training, even though they have had no occasion to use the analysis as an instrument of investigation. Many wish that they could use it better, as they see opportunities where it might be of advantage, if it were at their fingers' ends ; and all agree that, aside from its practical use, there is a training in habits of thought, points of view, and intellectual comprehension of ordinary engineering problems, which only the study of the higher mathematics can give. The fact is that the fundamental ideas of natural phenomena that appear in all technical work are found in their most logical and clearest form in the higher mathematical analysis. The development of the mind in logical relations whose analogues exist in some form or other in nature, creates that mental insight which perceives the dominant ideas and develops the most practical methods of treatment of the problems of each day.

The technical training that is involved in the study of the higher mathematics, particularly the calculus, therefore answers the question as to its value and utility for all engineering students, irrespective of any direct value of the analysis as an instrument of investigation.

II. Methods of Instruction.

Some practical men who look upon mathematical analysis only as an instrument of the engineer, would have it taught solely for the purpose of obtaining the requisite mechanical skill in its use. It is to be made a matter of memorizing formulas and methods, while any explanation is recommended that placates the mental comprehension of the student, whether it is logical and common sense or not, so long as it is brief. This is not only depriving the majority of students of the real benefits that accrue to them from the study of mathematics, such as in the acquirement of fundamental ideas of technical value, but it is also a dangerous method of instruction. When a young engineer can do something supposed to be of value in his profession, but about which he has only certain vague conceptions and no perfect understanding from a common sense point of view, it is quite natural that he should give it an exaggerated importance ; and by an equally obscure mental process, he obtains an exalted opinion of his own capabilities. The result is frequently disastrous to himself in some important work that he may have been led to undertake in the fancy that he was equal to it. The frequency of such cases, as coming from the graduates of a certain school, is certain evidence of faulty methods of instruction, particularly in mathematics. One difference between rigorous and logical methods of instruction, and memory training, is in their effect on what is called the personal equation of the student, *i. e.*, his liability to make mistakes. The first methods bring such equation before the student, show that he is in the habit of superficial thinking and of readily accepting plausible conclusions that are more often wrong than right. He is forced to develop a system of checks upon his ideas and conclusions, and to exercise a continual care in discovering and eradicating the effects of those lapses of mental acumen to which all are more or less liable according to the training of the individual and the temporary condition of the mind. On the other hand, training by memory alone leaves the individual in ignorance of his unconscious errors of thought, and while he may be very expert in his lessons, yet in the elements of detailed and accurate work, he may prove nothing but a blunderer, with all actual mental training to begin over again in practical life if he is to succeed.

Mathematical analysis should therefore be taught according to the precepts of good logic and common sense. It is not necessary to consider all the detailed refinement of the pure mathematician, but some of the consequences of that refinement should not be neglected, and above all the student should learn that some very good methods of demonstration for ordinary cases do have their exceptions. A combination of the recitation system, based upon a good text book, with the lecture system seems best, both because it secures that flexibility of course hereafter considered, and because a proper amount of lecture work is in itself a training in quickness of apprehension on the part of the student, and creates the ability to seize upon the important parts of a subject. In fact, extensive note-taking of lectures on the part of the student should not be encouraged, the idea being to work up the subject afterwards from as few notes as possible.

The working of examples, especially numerical ones, cannot be too excessive, if within the time of the student. In this connection, most students can, by proper thought given to methods of study and work, reduce the time necessary to be spent on that subject by fully one-half the usual amount spent unmethodically. This means, perhaps more time spent when beginning a subject, in learning thoroughly its fundamental principles and methods, and less time later on—just the reverse, it will be seen, of the usual procedure of the average student.

III. The Course.

With reference to the framework of a technical mathematical course, I should leave it as it is in most institutes —algebra, geometry, trigonometry, analytical geometry, the calculus, and the elements of differential equations with particular reference to those equations occurring in engineering subjects. Under geometry should be included the elements of projective geometry, especially of parallel projection, with applications to the conic sections. The calculus cannot be begun too early, before analytic geometry even. The calculus is the fundamental topic ; its ideas are so useful in mental development that when they are thoroughly mastered, other subjects are more easily acquired, so that it is in reality a time-saver. With us also, the last term of mathematical instruction is given up to the student's own devising in the way of subjects and development. Each student is expected to prepare what is essentially a course of lectures on chosen topics before the class. His aids in this work are whatever suggestions and information he can acquire from sources within his reach—the library, text books, his teachers. The result has been an improvement on knowledge already gained, and a development of pure and applied mathematics on the part of the student far beyond the usual course. Remembering that each student selects his own line of work, and hears, and is naturally interested in, whatever his classmates present, I have come to consider this term's work as the most valuable of all.

A course should be made flexible, by introducing at suitable times and places subjects that are interesting and likely to prove valuable in practical work. All pure mathematics is valuable as mental discipline; but the adaptability of different kinds will change with the years, the class, the character of the students, and the particular interests that they may be led by circumstances to manifest. For instance, I had, from a recent junior class, a request for a course in determinants, which is only briefly considered under algebra. The reason for this request was that the professor in dynamical engineering was at the time considering the subject of transformer problems, such as the problem of distribution of alternating current over a long line having capacity with up and down transformers at either end, and had found the determinant notation extremely useful in this work.

As another illustration of the changes which time brings, take the case of the applications of the imaginary analysis. The continuous current has had its day, although the practical applications of Faraday's discovery in 1832 was delayed nearly half a century by the want of a practical method of converting the alternating current into a continuous one. Recent discoveries have made the alternating current the more valuable in practical applications; and in this current we find a beautiful analogue of the imaginary analysis—so much so that technical writers are using that analysis in their publications on alternating currents for practical engineers. The imaginary analysis should therefore be taught to-day in live technical schools, with special reference to its geometrical and physical interpretations.

Instances might be multiplied showing that new circumstances will continually arise that necessitate deviations from old and established lines of instruction. Such flexibility in the course can only be secured, without continual change in its framework, through the lecture system in connection with text book instruction.

IV. The Instructor.

The success of a course, in mental training, and in meeting the practical needs of the engineer, depends very largely upon the instructor. He must be one who keeps up an interest in pure mathematics and their practical applications, and who is also clear in his own ideas, so that he may know what is interesting and useful, and be able to introduce it with clearness and brevity when occasion offers. He should, in other words, be a high grade man—not the highest, for that is impossible for all schools to secure, but certainly of such reputation among pure as well as applied mathematicians as causes him to be regarded with respect by the leaders in those subjects.

There is an objection in certain technical schools to a mathematical teacher who has too much fondness for pure mathematics. I once heard the president of such a school not only express this objection, but go further, by saying that he did not want a man in his faculty who knew more than himself on any subject.

The principal cause of the above objection to higher mathematical knowledge seems to be a fear that it leads to talking over the heads of the students. I will contend that it is an advantage to a school to have instructors who are able to talk over the heads of their students, and who do it occasionally. When an instructor talks over his head, the student may have some very strong feelings as to the futility of the exhibition, but he will find that the ideas do not always fly over. Like seeds planted in soil they await the germinating time. There is an unconscious impression made upon the mind by listening to ideas that are from a higher point of view than our own, and such impressions are often more durable than those left by apprehended impressions, and finally, as the result of gradual ascent to the higher point of view, they become fully understood and appreciated.

It is of course not possible to rely upon the results of such influences for the instruction of a technical student, and some ability to make himself understood and bring his mind within the confined horizon of the student, is necessary in the occupant of the instructor's chair. Such an instructor who also attempts to lift the students above their too small horizon at the certain risk of talking sometimes over their heads, produces, for some reason, students of higher acquisition, and of greater enthusiasm and practical ability, than is possible for the drill master to accomplish with his rule of thumb. It takes time to determine the influence of a teacher, and in the end the conscientious student is bound to recognize the superior influence upon his mental development of the instructor of superior attainments.

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