THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 23, 1901. Thirty-two persons were in attendance at the two sessions, including the following twenty-eight members of the Society:

Professor Anne L. Bosworth, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Professor T. S. Fiske, Mr. A. S. Gale, Mr. F. A. Giffin, Mr. Edwin Haviland, Jr., Dr. H. E. Hawkes, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. G. H. Ling, Dr. James Maclay, Dr. G. A. Miller, Mr. H. B. Mitchell, Professor E. H. Moore, Professor W. F. Osgood, Professor James Pierpont, Dr. M. B. Porter, Professor P. F. Smith, Professor Henry Taber, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor Eliakim Hastings Moore, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor John F. Downey, University of Minnesota, Minneapolis, Minn.; Professor Frederick C. Ferry, Williams College, Williamstown, Mass.; Mr. Henry T. Gerrans, Oxford University, Oxford, England; Mr. Edwin Haviland, Jr., New York, N. Y.; Professor Augustus E. H. Love, Oxford University, Oxford, England; Mr. V. R. Thyagarajaiyar, Bangalore, India. Two applications for membership were received. Action was taken by the Council toward making the Society's library accessible for the use of the members.

The following papers were read at this meeting:

(1) Dr. H. E. HAWKES: "Note on Hamilton's determination of irrational numbers."

(2) Professor E. B. VAN VLECK: "On the convergence of continued fractions with complex elements."

(3) Dr. M. B. PORTER: "On linear homogeneous finite difference equations, with applications to certain theorems of Sturm."

(4) Professor L. E. DICKSON: "Concerning real and complex continuous groups."

(5) Professor E. O. LOVETT: "An application of infinite groups to non-euclidean geometry."

(6) Professor E. O. LOVETT: "Contact transformations which change asymptotic lines into lines of curvature."

(7) Professor H. B. NEWSON: "Indirect circular transformations and mixed groups."

(8) Mr. W. B. FITE: "On metabelian groups that cannot be groups of cogredient isomorphisms."

(9) Dr. EDWARD KASNER: "On algebraic potential curves."

(10) Professor MAXIME BÔCHER: "Green's functions in space of one dimension."

(11) Dr. H. E. HAWKES: "Estimate of Benjamin Peirce's linear associative algebra."

(12) Dr. G. A. MILLER: "On holomorphisms and primitive roots."

(13) Dr. EDWARD KASNER: "Theorems on collinear lines in space."

(14) Mr. C. W. M. BLACK: "Decomposition of a form in n variables in an arbitrary domain with respect to a prime ideal modulus."

(15) Professor MAXIME BÔCHER: "An elementary proof of a theorem of Sturm."

(16) Dr. L. P. EISENHART: "Surfaces whose first and second fundamental forms are the second and first respectively of another surface."

(17) Dr. L. P. EISENHART: "Possible triply asymptotic systems of surfaces."

(18) Dr. H. F. STECKER: "On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves."

(19) Professor MAXIME BÔCHER: "Non-oscillatory linear differential equations of the second order."

Mr. Fite was introduced by Dr. G. A. Miller, and Mr. Black by Professor James Pierpont. In the absence of the authors, Professor Bôcher's first paper was read by Professor W. F. Osgood, and the papers of Professor Dickson, Professor Lovett, Professor Newson, Dr. Eisenhart, Dr. Stecker, and the second and third papers of Professor Bôcher were read by title.

Professor Newson's paper appeared in the March number of the BULLETIN. The present number contains the first paper of Professor Bôcher, the first paper of Dr. Hawkes. and the second paper of Dr. Eisenhart. The papers of Professor Dickson and Dr. Miller, and Professor Bôcher's second paper will appear in later numbers. The paper of Dr. Stecker and the third paper of Professor Bôcher will appear in the *Transactions*. Abstracts of the other papers are given below.

The paper of Professor E. B. Van Vleck, which will be published in the *Transactions*, gives first a résumé of the few existent theorems of a general character upon the convergence of continued fractions with complex elements. It then goes on to develop a set of equations which appears to be both new and fundamental for the study of continued fractions with complex elements. These equations are applied to the derivation of certain theorems, the first of which is as follows: If in a continued fraction

$$\frac{1}{\alpha_1 + \iota\beta_1} + \frac{1}{\alpha_2 + \iota\beta_2} + \frac{1}{\alpha_3 + \iota\beta_3} + \cdots$$

the β-constituents with even subscripts have one sign and those with odd subscripts the opposite sign, while the α-constituents have a common sign, the continued fraction will converge if $\sum_{n=1}^{\infty} \left| \alpha_n + \iota\beta_n \right|$ is divergent. On the other hand, if $\sum | \alpha_n + \iota\beta_n |$ is convergent, the even convergents approach one limit and the odd convergents another. It is further shown that neither the numerator nor the denominator of any one of the convergents of the continued fraction can vanish. This also holds true for the convergents of any continued fraction in which either the α_n or the β_n meet the conditions of the first theorem.

Continued fractions are next considered in which a finite number of the α-constituents or of the β-constituents (but not of both) fail to fulfill the conditions imposed in the first theorem, and such continued fractions are shown to converge. Under certain restrictions, also, an infinite number of exceptional elements may be admitted. The criterion for convergence stated in the first theorem accordingly differs from criteria hitherto obtained in that it admits of certain irregularities in the continued fraction whose positions therein are entirely arbitrary.

In conclusion, the results of the paper are applied to algebraic continued fractions. Thus it appears that if each element λ_n of a continued fraction

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots$$

is a positive number or such a number multiplied into z, the zeros of the numerators and denominators of the convergents will all lie upon the negative half of the z-axis. If, furthermore, no element with even (respectively with odd) subscript contains z and if $\sum |\lambda_n|$ is divergent, the continued fraction will converge over at least three-quarters of the plane; namely, over the half plane in which x is positive and in those sections in which x is negative and $|y| > |x|$. Within this region the continued fraction represents an analytic function and never vanishes. On the other hand, if $\sum |\lambda_n|$ is convergent, the even and odd convergents approach different limits, and these limits are likewise analytic and never vanish within the same region.

The theorems given by Sturm in his first important memoir in *Liouville's Journal*, Volume I., were first arrived at, so the author states at the end of the paper, as limiting cases of certain analogous theorems concerning linear difference equations of the second order. Sturm's work never having been published, Dr. Porter's paper aims at its restoration.

In the sixteenth volume of the *Acta Mathematica*, M. Zorawski followed a suggestion of the late Sophus Lie and determined the deformation invariants of surfaces in ordinary space by the method of the theory of continuous groups. It is the object of Professor Lovett's first paper to employ the same method to design the mechanism for the construction of the deformation invariants of varieties in a point space of any number of dimensions some power of whose lineal element is a homogeneous function of the differentials of the point coördinates of the space, and in particular to point out how the known and possible deformation invariants of surfaces in ordinary space and the theorems relative thereto may be extended immediately to surfaces in any quadratic space, euclidean or non-euclidean, of three dimensions. This particular result for varieties having three dimensions is especially interesting since it constitutes a contribution from the theory of groups to the geometry of all such varieties, and this in the face of the fact that the theory of finite groups is limited in its application to the construction of the geometries of three-dimensional manifoldnesses as is evidenced by the recent work of Bianchi (*Memoirs of the Italian Society of Sciences*, 3d series, vol. 11) and Cotton (thesis presented to the Faculty of Sciences of Paris), who have found all such varieties whose lineal elements admit of continuous groups.

The conclusion drawn follows from the fact that the deformation invariants of any particular class are determined once for all for every variety of three dimensions by the integration of one and the same formal system of linear partial differential equations. To construct this system of linear equations it is necessary to effect an m-fold extension of the infinitesimal transformations of an infinite group. Such an extension can be proposed in an infinite number of ways. The latter problem has also been studied by Zorawski (*Proceedings of the Cracow Academy of Sciences*, 2d series, vol. 4), who extends the transformation relative to the derivatives of its own functions; and by Levi-Civita (*Transactions of the Venetian Academy of Sciences, Letters and Arts*, 7th series, vol. 5), who extends the transformation relative to the elements of any covariant and contravariant systems whatever. The present note extends the transformation in three different ways, namely relative to all partial derivatives of any function whose variation is known, with regard to all derivatives of any one variable considered as a function of the other point coördinates, and finally with reference to the derivatives of a system of any number of invariant functions. The corresponding sys tems of differential equations give generalizations of the so-called deformation invariants of Gauss, Beltrami, and Minding.

The geometry of contact transformations calls for the determination of those transformations which change a surface into another surface in such a manner that a remarkable family of curves on one surface is transformed into a remarkable family of curves on the other surface. Such, for example, are those establishing correspondences between asymptotic lines, lines of curvature, Darboux lines, geodesic circles, etc. Those transforming asymptotic lines into such are known to be either projective or dualistic. This result is generalized in a subsequent note. Dilatations and inversions are known to change lines of curvature into lines of curvature. These theorems relative to asymptotic lines and lines of curvature are to be found in Darboux's Theory of Surfaces, volume I. It is proposed in Professor Lovett's second paper to determine those transformations which change asymptotic lines into lines of curvature. One such transformation was found by Sophus Lie, namely his celebrated correspondence between straight lines and spheres, in his well known memoir in the fifth volume of the *Mathematische Annalen*.

It appears that the functions X, Y, Z, P, Q, capable of defining a contact transformation possessing the desired property, must satisfy a certain system of equations of condition. The problem of determining the forms of the functions X, Y, Z, P, Q, from these general equations of condition offers serious difficulties. However, the means are at hand for the complete resolution of the problem, for Lie has remarked that all contact transformations which change asymptotic lines into lines of curvature also transform straight lines into spheres, and the latter transformations have been found by the author in a paper soon to be published. Accordingly, to find all contact transformations which change asymptotic lines into lines of curvature it is only necessary to subject the types of line sphere transformations already found to the equations of condition, a process simple but extremely laborious. It can be verified, however, without much difficulty that the functions X, Y, Z, P, Q determined by the sets of equations

$$\omega_1 = \sum_1^4 (a_{1j}X + ia_{1j}Y + a_{2j}Z + a_{3j})\, x_j = 0,$$

$$\omega_1 = \sum_1^4 (a_{2j}X - ia_{2j}Y - a_{1j}Z + a_{4j})\, x_j = 0,$$

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = 1,\ i = \sqrt{-1},$$

$$\omega_1^x/\omega_2^x = \omega_1^y/\omega_2^y = \omega_1^X/\omega_2^X = \omega_1^Y/\omega_2^Y$$

$$\omega_3 = \sum_1^4 a_{1j}X - ia_{1j}Y + a_{2j}Z + a_{3j})x_j = 0,$$

$$\omega_4 = \sum_1^4 (a_{2j}X + ia_{2j}Y - a_{1j}\,Y + a_{4j})x_j = 0,$$

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = 1, \quad i = \sqrt{-1},$$

$$\omega_3^x/\omega_4^x = \omega_3^y/\omega_4^y = \omega_3^X/\omega_4^X = \omega_3^Y/\omega_4^Y$$

constitute two families of ∞^{15} transformations which change asymptotic lines of a surface into the lines of curvature of the transformed surface. The celebrated transformation of Lie is included in both families ; we find it in the first by making

$$a_{21} = a_{42} = a_{33} = a_{14} = 1$$

and all the other constants zeros in the first set of equations, and also by putting

$$a_{11} = a_{32} = - a_{43} = - a_{24} = 1$$

and all the other coefficients zero in the second set of equations.

The above equations of condition can be simplified in several particulars. They reduce to fifteen in number if

$$B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = D_1 = D_2 = D_3 = 0,$$

and these last equations are readily found to be compatible. They are satisfied by Lie's transformation; moreover for the latter we have also

$$\begin{vmatrix} X^x X^y \\ Y^x Y^y \end{vmatrix} = \begin{vmatrix} X_p X_q \\ Y_p Y_q \end{vmatrix} = 0, \quad A_2^{\,2} - 4A_3A_1 = 0, \quad E_2^{\,2} - 4E_3E_1 = 0.$$

It appears incidentally that the determinant

$$\begin{vmatrix} X^y X_p & X^y Y_p + X_p Y^y & Y^y Y_p \\ X^x X_p - X^y X_q & X^x Y_p + X_p Y^x - X^y Y_q - X_q Y^q & Y^x Y_p - Y^y Y_q \\ X^x X_q & X^x Y_q + X_q Y^x & Y^x Y_q \end{vmatrix}$$

is zero for any contact transformation.

Mr. Fite shows 1° that a metabelian group of odd order which has a set of generators such that the order of one of them is not a divisor of the least common multiple of the orders of all the others cannot be the group of cogredient isomorphisms of any group; 2° that if a metabelian group of order p^a, where p is a prime, has a cyclical commutator subgroup and a group of cogredient isomorphisms with more than two independent generators, it can not be the group of cogredient isomorphisms of any group.

The curves considered in Dr. Kasner's first paper present themselves in the theories of functions, of equations, and of the potential. They are defined analytically by the vanishing of a rational integral solution of Laplace's equation $\varphi_{xx} + \varphi_{yy} = 0$; or, what is equivalent, by the vanishing of the real part of a rational integral function of $x + iy$. The paper gives geometric definitions of these potential curves, and some of their metric and projective properties. A potential curve is apolar to the degenerate class conic composed of the circular points at infinity; all of its polar conics are rectangular hyperbolas. Any curve of the nth order passing through the n^2 foci of a curve of the nth class is a potential curve; conversely, any potential curve passes through the foci of an infinite number of systems of confocal curves of the nth class. Conjugate potential curves intersect orthogonally in a set of such foci. This last result is applied to the theory of equations.

In 1870 Benjamin Peirce enumerated all systems of hypercomplex numbers of certain types in less than seven units. His principles of classification are five in number and are shown to be identical or closely related to the principles of classification suggested by the theory of continuous groups and adopted by Scheffers (*Mathematische Annalen,* vol. 39 (1891), p. 253). In Dr. Hawkes's second paper various criticisms of Peirce's work by Cayley, Study, and others are noted and discussed.

Dr. Kasner's second paper is in abstract as follows: Consider any five collinear lines L_1, L_2, L_3, L_4, L_5, *i. e.*, five lines having a common tractor M_0. In addition to M_0, any four of these lines as L_2, L_3, L_4, L_5 have another tractor M_1. It is proved in the first place that the five lines M_1, M_2, M_3, M_4, M_5, thus derived, are themselves collinear. Denote their common tractor by L_0. The remainder of the paper is devoted to the study of the configuration composed of the twelve lines L and M and of the thirty points and thirty planes which they determine. The same configuration is determined by any five of the lines L, or any five of the lines M. The anharmonic ratios of the various points and planes are connected by simple relations; all are expressible in terms of four, the fundamental absolute invariants. Finally these relations are employed to prove that the derived set M_1, M_2, M_3, M_4, M_5 is correlative to the original set L_1, L_2, L_3, L_4, L_5.

In Mr. Black's paper it was shown first that all coefficients of any possible factor can be brought within the residual system of the modulus, and then the irreducible factors can be determined by a finite number of trials. In the second part of the paper, the uniqueness of decomposition is shown, first for a single variable, and then by induction for n variables, the main part of the discussion being directed to bringing the problem into a form such that reasoning similar to that used for the proof of the uniqueness of algebraic decomposition may be applied.

Dr. Eisenhart's first paper is in abstract as follows: In view of the fact that there is a unique surface corresponding to two quadratic differential forms, whose coefficients satisfy the Codazzi and Gauss equations, the question is asked whether it is ever possible for the first and second fundamental forms of a surface to be the second and first respectively of a second surface. In answer it is found that there is a class of surfaces possessing this property ; that they are ruled surfaces with imaginary generatrices ; that the sphere of radius unity is the only real surface of the class ; and, furthermore, that the second surface differs from the first only by a translation in space.

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GREEN'S FUNCTIONS IN SPACE OF ONE DIMENSION.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, February 23, 1901.)

I WISH to make a brief communication to the Society of some results which I have obtained, reserving proofs and further developments for another occasion.

By a Green's function is ordinarily understood a solution of Laplace's equation which vanishes on the boundary of a certain region, and within this region is discontinuous at only one point, where it becomes infinite like $1/r$ or $\log r$, according as we are dealing with Laplace's equation in three or in two dimensions. We may speak of this as a Green's function of the first kind, in distinction to the generalized Green's function for which more complicated conditions than the mere vanishing of the function are imposed on the boundary of the region.

This fundamental conception has been generalized by replacing Laplace's equation on the one hand by other homogeneous linear partial differential equations of the second order (cf. Encyklopädie, volume 2, p. 516); on the other hand by the ordinary differential equation $d^2y/dx^2 = 0$, which may be called Laplace's equation in space of one dimension (cf. Burkhardt, *Bulletin de la Société mathématique de France*, volume 22 (1894), p. 71). This suggests at once the possibility of considering in place of Laplace's equation the general ordinary homogeneous linear differential equation of the second order. I find that we can not only do this,