$$x_{\alpha\beta} = \frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad y_{\alpha\beta} = \frac{y_\alpha + y_\beta}{2} - i\frac{x_\alpha - x_\beta}{2}.$$

As is well known, the n quantities

$$R(z_\alpha) \qquad (\alpha = 1, 2, \cdots, n),$$

where R denotes any rational function, satisfy an equation of the nth degree

$$F_n(t) = 0,$$

whose coefficients are rational in $A_0, A_1, \cdots, A_n$. This, however, no longer holds when we consider, instead of rational functions of the roots, rational functions of the real and imaginary parts of the roots; but if we consider the n^2 quantities

$$R(x_{\alpha\beta}, y_{\alpha\beta}),$$

they will satisfy an equation of the n^2 degree with coefficients which are rational in terms of the coefficients of φ and ψ, *i. e.*, in terms of $b_0, \cdots, b_n, c_0, \cdots, c_n$. Therefore,

THEOREM XV. *The n quantities*

$$R(x_1, y_1), \quad R(x_2, y_2), \cdots, R(x_n, y_n),$$

are the real roots of an equation of degree n^2 with coefficients which are rational in terms of the real and imaginary parts of the coefficients in (1) *; the remaining roots of the equation being*

$$R\left(\frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad \frac{y_\alpha + y_\beta}{2} + i\frac{x_\alpha - x_\beta}{2}\right), \quad (\alpha \neq \beta).$$

This result may easily be extended to functions of any number of roots $R(x_1, y_1, x_2, y_2, \cdots)$, and Theorem XIV may be extended to any system of simultaneous equations.

COLUMBIA UNIVERSITY,
February 25, 1901.

ALTERNATING CURRENT PHENOMENA.

Alternating Current Phenomena. By C. P. STEINMETZ. New York, Office of the Electrical World. Third Edition, 1900. Pp. xx + 525.

To electrical engineers Mr. Steinmetz's book is immediately conspicuous by reason of two distinguishing characteristics :

the first is the employment of a definite mathematical method of presentation, consistently maintained throughout the course of the work, and the second the employment of this method in the analysis of practically every problem in the application of alternating currents of electricity. A glance over the literature of applied electricity reveals no other work which stands forth so prominently in either of these characteristics, and the value of a treatment embracing them both can only be rightly estimated by those who have worked out their basic conceptions of alternating current phenomena and their applications from the conglomerate mass of trigonometry, differential equations, and inaccurate diagrams presented by earlier writers, and, found how inadequate it is for the solution of the problems confronting the engineer of today.

It is the first of these characteristics, namely the method of treatment that is the more interesting to mathematicians, and it is the purpose of this article to review the application of this method; a critical discussion of the complete work from the standpoint of the electrical engineer is not aimed at, and so all reference is omitted to much of the matter that is most valuable but hardly of interest in this place. Briefly stated, the method is the use of the algebra of the complex number in combination with a reference to polar coördinates of the alternating or periodic functions of current and electromotive force. A short consideration of a simple electric circuit carrying an alternating current will facilitate a review of the use to which this method has been put.

Threading or looping with a circuit carrying a current, there is a number of lines of magnetic force due to the passage of the current around the circuit; and this number rises, falls, and reverses with the varying values of the current; the induction or total number of these lines is thus also a periodically varying function "in phase" with the current. Due to this alternating field of force there is induced in the circuit an alternating electromotive force, which is shown by the law of Lenz to have its maximum one-quarter of the time of one complete period later than the inducing field, and so 90° later than that of the current, 360° representing a complete period; this is the counter electromotive force of self-induction. Due to the resistance of the conductor there is a consumption of electromotive force when the current flows, proportional to the current at each instant and so alternating and in phase with the current; this may be considered a counter electromotive force, 180° away from the current. If there is a condenser or electrical capacity in the circuit there is a third electromotive force which may be shown to have its phase 90° in advance of the current. It is the presence of these several angularly separated electromotive forces which causes the apparent failure of Ohm's law in the case of alternating current circuits. The impressed electromotive force necessary to cause the current to flow must overcome these several electromotive forces, *i. e.*, must have components angularly separated, and so in general will not be in phase with the current. It is to be noted that the difference in phase between current and counter electromotive force is either 0° or plus or minus 90°.

The fundamental principles suggested in the foregoing paragraph are assumed in the opening chapter of Mr. Steinmetz's book, as is also the form of the expression giving the value of the "impedance" or ratio of impressed E.M.F. to current. This expression for the impedance is $z = \sqrt{r^2 + x^2}$, r being the resistance or ratio of the in-phase component of the E.M.F. to the total current, and x the "reactance" or ratio of the out-of-phase component of the E.M.F. to the total current; since as indicated above there are two out-of-phase E.M.F.'s, one 90° in advance of, the other lagging 90° behind the current, x will take its value from the difference between the two E.M.F.'s due to self-induction and capacity, since they differ in phase by 180° or are opposed to each other. The values of x for self-induction and for capacity are calculated in terms of the frequency or number of periods per second and the constants of the circuit.

Passing now to chapter IV, the alternating or sinusoidal wave, represented by time as abscissa and instantaneous value as ordinate, is referred to polar coördinates, giving the circle as the curve; and for each complete period the circle is traversed twice, negative values of the function being taken when a reverse direction must be taken by the radius vector in order to intersect the curve. Thus the intercept on any radius vector gives the instantaneous value of the wave at the time represented by the amplitude of the vector. Since any particular value determines the wave, the step is then made of letting the diameter of the characteristic circle represent the wave, by its length the intensity and by its amplitude the phase; thus on the same diagram any number of E.M.F.'s and currents in a circuit differing in intensity and in phase may be represented by radii vectors of different lengths and amplitudes. The author here tacitly assumes that the current has at any instant the same value throughout the whole length of the circuit, *i. e.*, that the phase of the current wave does not change from point to point; cases in which this is unwarranted are considered in chapter XIII. The possibility of combining or resolving vectors of the same nature by the parallelogram law is then shown by considering combined instantaneous values on any chosen radius vector; thus the resultant of two electromotive forces, for instance, is represented by the diagonal of the parallelogram formed on the two radii representing their intensities and phases. The graphical method here clearly developed gives perhaps the clearest insight possible into the mutual relations of the several alternating sine waves entering into any problem.

Owing to the widely differing magnitudes of the alternating waves to be represented in the same diagram, the graphical method is not well suited for numerical calculation, and in chapter V the author extends the graphical treatment into the symbolic method, which, instead of denoting the vector representing the sine wave by the polar coördinates of intensity I and amplitude ω, uses the rectangular coördinates $a = I \cos \omega$ and $b = I \sin \omega$, thus avoiding the use of trigonometric functions in the combination or resolution of waves. Extending the relations of the parallelogram law: "Sine waves are combined or resolved by adding or subtracting their rectangular components." To distinguish between the two components, the symbol j is put before the vertical component, $I = a + jb$, meaning that a is the horizontal and b the vertical component of the wave I and that they are combined in the wave of resultant intensity $i = \sqrt{a^2 + b^2}$; similarly $a - jb$ is a wave with a as horizontal and $-b$ as vertical component. The next step brings in the full significance of the method; multiplying the symbolic expression $a + jb$ by -1 evidently gives $-a - jb$, or a wave of equal intensity but differing in phase by 180°; a wave of equal intensity but lagging in phase by 90° (clockwise rotation) is evidently represented by $ja - b$; this expression may be derived by multiplying the expression $a + jb$ by j if upon the "hitherto meaningless symbol j" the condition be imposed $j^2 = -1$; and similarly multiplying by $-j$ advances the wave through 90° or one quarter of a period. The symbol j is thus seen to be the imaginary unit, and the sine wave is represented by the complex quantity of the type $a + jb$; the letter j is used instead of the usual i, since the latter so commonly in electrical literature denotes the current. "As the imaginary unit j has no meaning in the system of ordinary numbers, this definition of $j = \sqrt{-1}$ does not contradict its original introduction as a meaningless symbol.'' Thus $a + jb$ means a wave of intensity $i = \sqrt{a^2 + b^2}$ and of phase $\omega = \tan^{-1}\frac{b}{a}$; it may also be represented by $i(\cos\omega + j\sin\omega)$ and also by $i\varepsilon^{j\omega}$. A further extension of method now gives: ''Sine waves may be combined or resolved by adding or subtracting their complex algebraic expressions.'' The complex expression for the impedance is then developed and may here serve as a simple instance of the use of the method. A current $I = i + ji'$ flows in a circuit of resistance r and reactance x; the E.M.F. consumed by resistance is in phase with the current and is $rI = ri + jri'$: this E.M.F. must be supplied by the impressed E.M.F., as must also be an E.M.F. necessary to overcome the counter E.M.F. due to the reactance x; this E.M.F., if it be due to self induction, lags 90° behind the current and is therefore represented by $jxI = jxi - xi'$ (if x be due to capacity, by $-jxI = xi' - jxi$); the component of the impressed E.M.F. to overcome this is evidently $-jxI = -jxi + xi'$; and the E.M.F. to overcome both resistance r, and reactance x is $E = (r - jx) I$; or, the ratio of electromotive force to current, *i. e.*, the impedance, has for its complex expression $Z = r - jx$. The relation $E = ZI$, the complex values of the three quantities being used, may be handled in any of its three forms, giving a simple complex expression for any one of the quantities in terms of the components of the other two, since the imaginary may be easily eliminated from a denominator, and thus the real and imaginary components separated.

From the foregoing, it is seen that the total impedance of a circuit, consisting of any number of portions differing as to r and x, and connected in series, may be obtained by adding the complex expressions for the impedance of the several portions. The total impedance being known, the E.M.F. necessary to supply any given value of the current to such a circuit may at once be had, together with its phase relation to the current. This is the general nature of one class of problems. If, however, the circuit has several branches, or if several currents are supplied by the same E.M.F., the total impedance of the circuit is not a simple expression, just as the resistance of a number of branches connected in parallel is not a simple expression in the resistances of the several branches. In the latter case, however, the joint conducting power, *i. e.*, the reciprocal of the resistance, is the sum of the ''conductances'' of the several branches. So for parallel connected branches the author derives in chapter VII a method for combining, by the addition of complex quantities, the effects of the several impedances of a branched circuit. The total current supplied by the impressed E.M.F. is the sum of the currents in the several branches, attention being paid to their phase relations. *i. e.*, the complex expressions of the currents are to be added. To simplify this, Ohm's law, which now holds if complex expressions are used, is put in the form $I = EY$, where Y, being the reciprocal of Z, is a complex quantity. and the values of I for the several branches are now readily added. Y is called the admittance, and the total admittance of a branched circuit is the sum of the complex expressions of the individual admittances. Y, being complex, is of the form $g + jb$. We have

$$Y = g + jb = \frac{1}{Z} = \frac{1}{r - jx} = \frac{r + jx}{r^2 + x^2}.$$

Therefore

$$g = \frac{r}{r^2 + x^2} \quad \text{and} \quad b = \frac{x}{r^2 + x^2},$$

so that the expressions for Y and Z for any branch or combination of branches are readily derived one from the other.

Chapters VIII and IX give a complete investigation of the various types of series and parallel circuits, of the effects upon regulation and phase difference of the relative values of r, x, g, and b; and results heretofore obtained only in very complicated form are reduced to simple algebraic expressions. A single simple case will suffice here, but no electrical engineer should fail thoroughly to digest the contents of these two chapters.

A reactance x_0 is inserted in series with a load circuit of impedance $Z = r - jx$, and an E.M.F. E impressed upon the whole. The total impedance is $Z - jx_0$ or $r - j(x + x_0)$; the current is

$$I = \frac{E_0}{r - j(x + x_0)},$$

with the absolute value $\dfrac{E_0}{\sqrt{r^2 + (x + x_0)^2}} = \dfrac{E_0}{\sqrt{z^2 + 2xx_0 + x_0^2}}$;

the E.M.F. on the receiver or load circuit is

$$E = IZ = \frac{E_0(r - jx)}{r - j(x + x_0)} = E_0 \sqrt{\frac{r^2 + x^2}{r^2 + (x + x_0)^2}}$$

$$= \frac{E_0 z^2}{\sqrt{z^2 + 2xx_0 + x_0^2}}.$$

Generally in such a case the value of E, as compared with E_0, is of prime importance, and the above expression gives means of controlling E by a proper variation of x_0; thus $E = E_0$ if $x_0 = -2x$; if $x_0 < -2x$ it raises, if $x_0 > -2x$ it lowers the voltage; if x and x_0 have the same sign E is always less than E_0. It is to be remembered that a positive value of x is given by self-induction, a negative value by capacity. The difference in phase between current and E.M.F. is gotten, as indicated above, from the expression for the impedance; here the difference in phase in the load circuit is $\omega = \tan^{-1}\frac{x}{r}$; and in the supply or generator circuit $\omega' = \tan^{-1}\frac{x + x_0}{r}$.

Chapter XII is new in the third edition; in it the author attempts to extend the symbolic method to quantities of double frequency, such as the power. At any instant the flow of power in an alternating current circuit is the product of the instantaneous values of current and E.M.F. If two sine waves, *e. g.*, one of current and one of E.M.F. differing in phase, be drawn in rectangular coördinates, and if also a curve representing the product of their instantaneous values be drawn, it is found that while either the current or E.M.F. has passed through half a period, the curve of products, or the power wave, has passed through a complete period; that is, the power has double the frequency of the current and E.M.F., and so may not be represented on the same vector diagram with them. The area of positive values in the power curve represents power given into the medium by the circuit, that of negative values power returned to the circuit from the medium, this power having been stored there in the forms of a magnetic field and an electrostatic strain; the difference is the true expenditure of power. The product of the complex expressions for current and E.M.F., $(e' + je'')(i' + ji'') = (e'i' - e''i'') + j(e''i' + e'i'')$, does not represent the power, since it is an expression of the same frequency as the current and E.M.F.; suppose, however, since the power is of double frequency, the phase angle be doubled in the above expression; *i. e.*, instead of $j^2 = -1$, corresponding to a rotation through 180°, we now have $j^2 = +1$, or 360° rotation, and multiplication by j merely reverses the sign or rotates through 180°. The product then becomes $(e'i' + e''i'') + j(e''i' - e'i'')$, the first term of which is the real power $EI \cos \omega$, and the second the author calls the "wattless power," or $EI \sin \omega$. The chapter is interesting in explaining the apparent failure of the product of the two complex expressions to represent the power, but is unattractive, except to those versed in non-commutative algebra, because of the necessity of remembering that $j \times 1 = j$ is not the same as $1 \times j = -j$.

Chapter XIII is devoted to those cases where it is not permissible to assume that at any instant the value of the current is the same throughout the circuit; an instance is a submarine cable or any line along which capacity in some quantity is uniformly distributed. In such cases the simple vector diagram and the algebra of complex quantities do not suffice; however, by considering the values of r, x, g, b per unit length of line, the author forms and solves the differential equations for both current and E.M.F., as varying from point to point in the line. While exhaustive in discussion and most useful, the chapter offers no striking application of the symbolic method.

In chapters XIV, XV, and XVI the symbolic method is extended to the analysis of the transformer and the induction motor, *i. e.*, the motor with rotating magnetic field; they are shown to belong to the same general type of apparatus (a fact not before recognized), called by the author the general alternating current transformer. Consider the simple transformer consisting of a magnetic circuit interlinked with two electric circuits, a primary and a secondary. The primary circuit carrying current sets up a field in the magnetic circuit, which induces an E.M.F. in the secondary which supplies current to its load. The secondary is now considered as a simple circuit with a given impressed E.M.F., having an internal impedance $Z_1 = r_1 - jx_1$ due to its resistance and self-induction, and feeding a load circuit of impedance $Z = r - jx$. Since the same magnetic circuit links with both coils, the actions in the secondary are shown to be reducible to the primary by the ratio of the numbers of turns in each; the primary has also its internal impedance $Z_0 = r_0 - jx_0$; combining the primary impedance with the reduced values of the secondary circuit, the effect of the whole transformer is brought to the expression of a single impedance. By proper attention to the difference in frequency between primary and secondary due to the "slip" of the armature (secondary) behind the rotating field due to the primary, the same method of procedure is adopted for the induction motor.

Chapter XXIV is a most interesting extension of the symbolic method to the representation of the general alternating wave as distinguished from the simple sine wave of the type $A = a_0 \cos(\varphi - \alpha)$; to the latter only is the vector representation $A = a' + ja'' = a(\cos\alpha + j\sin\alpha)$ applicable.

If the two half periods of a wave are similar, the even harmonics are absent and the general wave is expressed by

$$A = A_1 \cos(\varphi - a_1) + A_3 \cos(3\varphi - a_3) \\ + A_5 \cos(5\varphi - a_5) + \cdots,$$

which may not be represented by a single complex vector quantity. The individual harmonics, however, of this general wave are independent and no products appear, so that each may be represented by a complex symbol and the symbolic expression for the general wave is

$$A = \sum_1^\infty (2n-1)(a_n' + j_n a_n'');$$

here $j_n = \sqrt{-1}$ always, but the index of j_n denotes that the j's of different indices, while equal algebraically, physically represent different frequencies and so cannot be combined. The general wave of E.M.F. is thus represented by

$$E = \sum_1^\infty (2n-1)(e_n' + j_n e_n''),$$

and the current by a similar expression in the i's. The expression for the impedance undergoes some change; the values of x, the reactance, when due to self-induction, are directly, when due to capacity, inversely proportional to the frequency; there is also a part independent of the frequency; thus the impedance of a circuit will have different values for the several harmonics and its general expression is

$$Z = r - j_n\left(nx_m + x_0 + \frac{x_c}{n}\right).$$

Operations according to Ohm's law may now be performed on the general wave quantities E, I, and Z, just as on the simple sine wave; multiplication and division, however, being only performed on those terms having the same index n.

A common and most useful medium for the use of alternating currents is the so called polyphase system in which several equal E.M.F.'s differing in phase by the same angle are generated in the same machine; the induction motor is the most conspicuous form of apparatus depending on the polyphase system. The symbolic method lends itself admirably to the representation of such a system by means of the nth roots of unity. In the polar diagram the n E.M.F.'s of an n-phase system are represented by n equal vectors following each other under equal angles $2/\pi n$. In symbolic notation, advance or rotation through an angle $2\pi/n$ is represented by multiplying by the quantity $\cos 2\pi/n + j \sin 2\pi/n$, and so the E.M.F.'s of a polyphase system are

$$E, \quad E\left(\cos\frac{2\pi}{n} + j\sin\frac{2\pi}{n}\right), \quad E\left(\cos\frac{4\pi}{n} + j\sin\frac{2\pi}{n}\right), \text{ etc.}$$

In chapters XXVI and XXVIII the author handles this application for the deduction of the expression for the rotating magnetic field, the ring and star E.M.F.'s of interlinked systems, and other matters of general use, but up to this time wanting an analytical expression.

The writer has only gratitude to express at the appearance of this work, and his one regret is that its author did not also include in it his recent articles on the rotary converter.

JOHN B. WHITEHEAD, JR.

JOHNS HOPKINS UNIVERSITY,
April 18, 1901.

SHORTER NOTICE.

Leçons Nouvelles sur les Applications Géométriques du Calcul Différentiel. By W. DE TANNENBERG. Paris, A. Hermann, 1899. 192 pp.

THIS volume, which M. de Tannenberg has contributed to the literature of the theory of curves and surfaces, is very opportune. We have wanted a book which would make possible for the beginner a knowledge of the more fundamental geometrical applications of the calculus and in a way which would prepare him for the treatises of Darboux and Bianchi. To be sure, this field has been covered, more or less, in the chapters devoted to geometrical applications in the French treatises on analysis—notably by Jordan, Picard, Appell—but rather as examples of the methods of analysis and not standing forth as a systematic development of the elements of another field of mathematics. Again, there have been in recent years, quite a number of shorter treatises with just the scope of the volume under discussion, but their treatment of the subject has been along lines quite different from the well known methods of the calculus: Ricci in his Lezioni arrives at the results by the study of