

CONCERNING SURFACES WHOSE FIRST AND
SECOND FUNDAMENTAL FORMS ARE THE
SECOND AND FIRST FUNDAMENTAL
FORMS RESPECTIVELY OF
ANOTHER SURFACE.

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IN the July number of the BULLETIN Dr. Eisenhart treated these surfaces and reached the following conclusions: "The ruled surfaces defined by the equations

$$y + \mu x = \sqrt{1 + \mu^2} + C_1 \mu + C_1$$

$$z - ix \sqrt{1 + \mu^2} = (\mu + C_1 \sqrt{1 + \mu^2} + C_3) \cdot \frac{1}{i}$$

are the only surfaces whose first and second fundamental forms can be taken for the second and first fundamental forms of a surface. Further the second surface is only the first to a translation près. And of these surfaces the only real one is the sphere of radius unity."

In fact the above equations represent the spheres of radius unity and nothing else. For we have

$$y + zi + (\mu + \sqrt{1 + \mu^2})x = (c_1 + 1)(\mu + \sqrt{1 + \mu^2}) + c_2 + c_3$$

$$y - zi + (\mu - \sqrt{1 + \mu^2})x = (c_1 - 1)(\mu - \sqrt{1 + \mu^2}) + c_2 - c_3$$

or

$$(y - c_2) + (zi - c_3) = -(x - c_1)(\mu + \sqrt{1 + \mu^2}) + (\mu + \sqrt{1 + \mu^2})$$

$$(y - c_2) - (zi - c_3) = -(x - c_1)(\mu - \sqrt{1 + \mu^2}) - (\mu - \sqrt{1 + \mu^2}),$$

i. e.,

$$(y - c_2)^2 + (z + c_3 i)^2 + (x - c_1)^2 = 1.$$

But there are two more surfaces of the kind, both imaginary, both spheres of the radii ω , ω^2 , respectively, where $\omega = \sqrt[3]{1}$.

For let

$$x = R \sin u \cos v, \quad y = R \sin u \sin v, \quad z = R \cos u.$$

Then

$$\begin{aligned} E &= R^2, & G &= R^2 \sin^2 u, & F &= 0, \\ D &= R, & D' &= 0, & D'' &= R \sin^2 u. \end{aligned}$$

If now (1) $R = \omega$,

$$E_\omega = \omega^2, \quad G_\omega = \omega^2 \sin^2 u, \quad D_\omega = \omega, \quad D_\omega'' = \omega \sin^2 u;$$

and if (2) $R = \omega^2$,

$$E_{\omega^2} = \omega, \quad G_{\omega^2} = \omega \sin^2 u, \quad D_{\omega^2} = \omega^2, \quad D_{\omega^2}'' = \omega^2 \sin^2 u.$$

No other surface of the kind exists. For, as is shown in the article referred to, for these surfaces (ρ_1 and ρ_2 being two principal radii of curvature)

$$\rho_1 = \rho_2 = \frac{E}{D} = \frac{G}{D''}$$

i. e.,

$$\frac{D}{E} = \frac{D''}{G} = \lambda.$$

Bianchi has shown (*Differentialgeometrie*, German translation, p. 93, footnote), that λ is constant, if D, D', D'', E, F, G are proportional.

Using the equations (1) and (9) of Dr. Eisenhart's article, we have

$$\begin{aligned} \lambda^2 \cdot \sqrt{EG} + \frac{\partial}{\partial u} \left[\frac{1}{\sqrt{E}} \cdot \frac{\partial G}{\partial u} \right] + \frac{\partial}{\partial v} \left[\frac{1}{\sqrt{G}} \cdot \frac{\partial E}{\partial v} \right] &= 0, \\ \frac{\sqrt{EG}}{\lambda} + \frac{\partial}{\partial u} \left[\frac{1}{\sqrt{E}} \cdot \frac{\partial G}{\partial u} \right] + \frac{\partial}{\partial v} \left[\frac{1}{\sqrt{G}} \cdot \frac{\partial E}{\partial v} \right] &= 0. \end{aligned}$$

Hence, on subtraction,

$$\lambda^2 \sqrt{EG} = \frac{\sqrt{EG}}{\lambda}, \quad \lambda^3 = 1,$$

i. e.,

$$\lambda = 1, \quad \omega, \quad \omega^2.$$

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