

$$(22) \quad z_i P(y) = \frac{d}{dx} \left[ (-1)^{i-1} \frac{W(y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n)}{W(y_1, \dots, y_n)} \right],$$

where  $y_1, \dots, y_n$  are any set of  $n$  linearly independent solutions of (19) and  $z_1, \dots, z_n$  are the functions adjoint to them. Thus we have proved the theorem

XVIII. *A necessary and sufficient condition that  $z$  be a multiplier of (19) is that it be a member of the linear family adjoint to the family which consists of the solutions of (19).*

GRUND IM HARZ,  
July 20, 1901.

## THE CONFIGURATIONS OF THE 27 LINES ON A CUBIC SURFACE AND THE 28 BITANGENTS TO A QUARTIC CURVE.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, August 20, 1901.)

### *Introduction.*

AFTER determining\* four systems of simple groups in an arbitrary domain of rationality which include the four systems of simple continuous groups of Lie, the writer was led to consider the analogous problem for the five isolated simple continuous groups of 14, 52, 78, 133, and 248 parameters. The groups of 78 and 133 parameters are related to certain interesting forms of the third and fourth degrees respectively.† They suggested the forms  $C$  (§ 1) and  $Q$  (§ 3).

It is shown in § 1 that the cubic form  $C$  defines the configuration of the 27 straight lines on a cubic surface in or-

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the functions  $y_1, \dots, y_k$  being supposed to be any functions of  $x$  which throughout ( $J$ ) have continuous derivatives of the first  $k-1$  orders. This establishes the truth of ( $F$ ) at all points of ( $J$ ) except where

$$W(y_1, \dots, y_{m-1}) = 0.$$

If  $c$  is a point where this last equality holds two cases are possible: 1° there may be points in every neighborhood of  $c$  where the equality does not hold and where therefore ( $F$ ) holds. In this case, on account of the continuity of both sides of ( $F$ ), this formula holds also at  $c$ . 2°  $W(y_1, \dots, y_{m-1})$  may vanish identically throughout the neighborhood of  $c$ . In this case ( $F$ ) also holds at  $c$  since all the Wronskians which occur in it vanish at  $c$ ; cf. *Transactions*, vol. 2, p. 148.

\* Abstract presented to the Society, Aug. 20, 1901, to appear in extent in the *Transactions*. A note on the subject appeared in *Comptes rendus*, CXXXII. (1901), pp. 1547-8.

† Cartan, Thèses, Paris, 1894.

dinary space. After proving this result, the writer observed that the formulæ remained unaltered if the notation for the variables was chosen to be  $x_i, y_i, z_{ij} \equiv z_{ji}$  ( $i, j = 1, \dots, 6; j \neq i$ ), a notation given by Burkhardt.\* The notation (1) has been retained in view of the relation with the later sections and to retain uniformity with the notation of a paper † on the transformation group defined by the invariant  $C$  for an arbitrary domain of rationality.

The group of the configuration of the 27 lines on a cubic surface is exhibited in § 2. A study of the quartic form  $Q$  and the group of the configuration defined by it is made in §§ 3-6.

§ 1. *The 27 Lines on a Cubic Surface.*

A general cubic surface contains 27 straight lines such that

1°. Any one of the 27 lines  $A$  meets 10 other lines which intersect by pairs, forming with  $A$  5 triangles. The total number of such triangles on the surface is therefore 45.

2°. Any two triangles  $ABC$  and  $A'B'C'$  having no side in common determine uniquely a third triangle  $A''B''C''$ , such that the corresponding sides of the three triangles intersect and form three new triangles  $AA'A'', BB'B'', CC'C''$ . The former set of three triangles is said to constitute a *trieder*. Each triangle lies in exactly 16 trieders.

These two properties completely define the configuration of the 45 triangles formed by the 27 lines on the cubic surface.

We employ the 27 variables

$$(1) \quad x_i, y_i, z_{ij} \equiv -z_{ji} \quad (i, j = 1, \dots, 6; j \neq i)$$

and consider the cubic form with 45 terms ‡

$$C \equiv \sum_{i+j}^{i, j=1, \dots, 6} x_i y_j z_{ij} + \sum z_{\lambda\mu} z_{\nu\rho} z_{\sigma\tau}$$

the second sum comprising the 15 terms of the Pfaffian [1 2 3 4 5 6], so that the subscripts have the following values:

$$\begin{array}{lll} 12 \ 34 \ 56, & 13 \ 24 \ 65, & 14 \ 23 \ 56, \\ 12 \ 35 \ 64, & 13 \ 25 \ 46, & 14 \ 25 \ 63, \\ 12 \ 36 \ 45, & 13 \ 26 \ 54, & 14 \ 26 \ 35, \end{array}$$

\* *Math. Annalen*, vol. 41, p. 339.

† Offered by the writer, July 18, 1901, to the *Quar. Jour. of Math.*

‡ Derived from the function  $J$  of Cartan (l. c., p. 143) upon replacing each  $y_j$  by  $-y_j$ . The character of  $J$  was not considered by Cartan.

$$\begin{aligned} &15\ 23\ 64, \quad 16\ 23\ 45, \\ &15\ 24\ 36, \quad 16\ 24\ 53, \\ &15\ 26\ 43, \quad 16\ 25\ 34. \end{aligned}$$

Let the 27 lines on the cubic surface be represented by the variables (1), so that  $z_{ji} \equiv -z_{ij}$  represents the same line that  $z_{ij}$  represents. We proceed to prove that *the arrangement of the 27 variables into the 45 triples given by the terms of C furnishes a suitable notation for the configuration of the 45 triangles formed by the 27 straight lines on a general cubic surface in ordinary space.*

Of the triples exhibited by the terms of C, exactly 5 contain  $x_i$ ; exactly 5 contain  $y_j$ ; exactly 5 contain  $z_{ij}$ ; viz.,

$$x_j z_{ij}, \quad x_j z_{ji}, \quad z_{ij} z_{mn} z_{pq}, \quad z_{ij} z_{mp} z_{qn}, \quad z_{ij} z_{mq} z_{np},$$

where  $i, j, m, n, p, q$  form a permutation of 1, 2, 3, 4, 5, 6.

For  $i$  and  $j$  fixed, the triple  $x_j z_{ij}$  lies in exactly 16 triaders

$$\left\{ \begin{matrix} x_i & y_j & z_{ij} \\ z_{ki} & x_k & y_i \\ y_k & z_{jk} & x_j \end{matrix} \right\} \text{ (four), } \left\{ \begin{matrix} x_i & y_j & z_{ij} \\ y_k & x_l & z_{lk} \\ z_{ik} & z_{lj} & z_{rs} \end{matrix} \right\} \text{ (twelve),}$$

where  $i, j, k, l, r, s$  form an even permutation of 1, 2, 3, 4, 5, 6.

The triple  $z_{12} z_{34} z_{56}$  lies in 12 triaders of the type just given and in exactly four additional triaders

$$\left\{ \begin{matrix} 12 & 34 & 56 \\ 35 & 26 & 14 \\ 64 & 15 & 23 \end{matrix} \right\}, \left\{ \begin{matrix} 12 & 34 & 56 \\ 53 & 16 & 24 \\ 46 & 25 & 13 \end{matrix} \right\}, \left\{ \begin{matrix} 12 & 34 & 56 \\ 36 & 15 & 24 \\ 54 & 26 & 13 \end{matrix} \right\}, \left\{ \begin{matrix} 12 & 34 & 56 \\ 63 & 25 & 14 \\ 45 & 16 & 23 \end{matrix} \right\}.$$

Similarly,  $z_{ik} z_{ij} z_{rs}$  lies in exactly 16 triaders. Hence the notation is suitable to exhibit the configuration.

§ 2. Group G of the Equation for the 27 Lines.

The group G of the configuration of the 45 triangles formed by the 27 lines on a general cubic surfaces is composed of the literal substitutions on the variables (1) which leave the function C invariant. The determination of G has been effected by Jordan (Traité, pp. 316-329) and by the writer (Linear groups, Chapter XIV.). The notations of the latter treatment may be identified with the present notations as follows :

$$\begin{aligned} x_1 &= R_0, & x_2 &= R_{212}, & x_3 &= R_{111}, & x_4 &= R_{121}, & x_5 &= R_{131}, & x_6 &= R_{141}, \\ y_1 &= R_{112}, & y_2 &= R_{11}, & y_3 &= R_{210}, & y_4 &= R_{120}, & y_5 &= R_{130}, & y_6 &= R_{140}, \\ z_{12} &= R_2, & z_{18} &= R_{110}, & z_{14} &= R_{220}, & z_{15} &= R_{230}, & z_{16} &= R_{240}, \\ z_{23} &= R_{211}, & z_{24} &= R_{221}, & z_{25} &= R_{231}, & z_{26} &= R_{241}, & z_{34} &= R_{222}, \\ z_{35} &= R_{232}, & z_{36} &= R_{242}, & z_{45} &= R_{142}, & z_{46} &= R_{132}, & z_{56} &= R_{122}. \end{aligned}$$

It follows that  $G$  is generated by the substitutions

$$\begin{aligned}
 W : & \quad (x_1 y_2 z_{12})(x_2 y_3 z_{23})(x_3 y_1 z_{13})(y_4 x_4 z_{66})(y_5 x_5 z_{46})(y_6 x_6 z_{45}) \\
 & \quad (z_{14} z_{24} z_{34})(z_{15} z_{25} z_{35})(z_{16} z_{26} z_{36}); \\
 E_1 : & \quad (x_4 y_5 z_{26})(y_5 y_4 z_{13})(x_6 z_{24} z_{25})(x_2 z_{46} z_{66})(y_1 z_{35} z_{34})(y_3 z_{15} z_{14}); \\
 E_2 : & \quad (y_5 y_6)(y_4 z_{13})(y_3 z_{14})(y_1 z_{34})(x_2 z_{56})(x_5 z_{25})(x_6 z_{26}) \\
 & \quad (z_{15} z_{16})(z_{35} z_{36})(z_{45} z_{46}); \\
 E_3 : & \quad (y_2 y_6)(y_4 z_{13})(y_3 z_{14})(y_1 z_{34})(x_2 z_{25})(x_5 z_{26})(x_6 z_{56}) \\
 & \quad (z_{12} z_{16})(z_{23} z_{36})(z_{24} z_{46}); \\
 T : & \quad (y_4 y_6)(x_4 x_6)(z_{14} z_{16})(z_{24} z_{26})(z_{34} z_{36})(z_{54} z_{56}).
 \end{aligned}$$

Of these,  $W$ ,  $E_1$ ,  $E_2$ , and  $E_3$  give rise to even substitutions on the 45 triangles, while  $T$  gives rise to an odd substitution. The group  $G$  of order 51840 therefore has a subgroup  $G_{25920}$  generated by  $W$ ,  $E_1$ ,  $E_2$ ,  $E_3$ . A suitable product of the latter replaces  $x_i$  by an arbitrary one of the 27 variables. Hence  $G_{25920}$  is transitive.

§ 3. Character of the Quartic Form  $Q$ .

We employ the 56 variables

$$(2) \quad x_i, y_i, x_{ik} = -x_{ki}, y_{ik} = -y_{ki} \quad (i, k = 1, \dots, 7; k \neq i)$$

and consider the quartic form with 630 terms\*

$$Q \equiv \sum x_i y_j x_{ik} y_{jk} + \sum x_{\lambda\mu} x_{\nu\rho} y_{\mu\nu} x_{\lambda\rho} + \sum (x_i y_{\lambda\mu} y_{\nu\rho} y_{\sigma\tau} + y_i x_{\lambda\mu} x_{\nu\rho} x_{\sigma\tau}),$$

where in the first sum  $i, j, k = 1, \dots, 7; i \neq j, i \neq k, j \neq k$ ; in the second sum  $\lambda, \mu, \nu, \rho$  run through the various permutations of 1, 2, ..., 7 four at a time, but taking only one of the four equal terms

$$x_{\lambda\mu} x_{\nu\rho} y_{\mu\nu} y_{\lambda\rho} = x_{\mu\lambda} x_{\rho\nu} y_{\lambda\rho} y_{\mu\nu} = x_{\nu\rho} x_{\lambda\mu} y_{\rho\lambda} y_{\nu\mu} = x_{\rho\nu} x_{\mu\lambda} y_{\nu\mu} y_{\rho\lambda};$$

in the third sum  $i, \lambda, \mu, \nu, \rho, \sigma, \tau$  denotes an even permutation of 1, 2, ..., 7, so that the coefficient of  $x_i$  is the Pfaffian  $[1 \ 2 \ \dots \ i - 1 \ i + 1 \ \dots \ 7]$  defined in § 1.

Each of the three sums in  $Q$  extends over 210 terms. The fact that there are 630 terms in  $Q$  also follows from the lemma next proved, since  $\frac{1}{2} 45 \times 56 = 630$ .

LEMMA.—Each variable lies in exactly 45 terms of  $Q$ . The complementary cubic factor defines the configuration of the 45 triangles formed by the 27 straight lines on a cubic surface.

\* Suggested by the essentially different function  $J$  of Cartan (p. 144).

There are 30 and 15 terms respectively in the sums

$$\frac{\partial Q}{\partial x_i} \equiv \sum y_j x_{ik} y_{jk} + \sum y_{\lambda\mu} y_{\nu\rho} y_{\sigma\tau},$$

where  $j, k = 1, \dots, 7; j \neq i, k \neq i, j \neq k$ ; and  $\lambda, \mu; \nu, \rho; \sigma, \tau$  run through the subscripts of the 15 terms of the Pfaffian  $[1, 2, \dots, i - 1, i + 1, \dots, 7]$ . To identify  $\frac{\partial Q}{\partial x_i}$  with the function  $C$  of § 1, we have only to set  $x_{ik} = -x_k, y_{jk} = z_{jk}$ . From  $\frac{\partial Q}{\partial x_i}$  we obtain  $\frac{\partial Q}{\partial y_i}$  by interchanging  $y_j$  with  $x_j, y_{jk}$  with  $x_{jk}$ , an operation defining a substitution  $A$  which leaves  $Q$  unaltered. Similarly (§ 4) there exist substitutions which leave  $Q$  invariant and replace  $x_i$  by any one of the 56 variables. Hence the lemma holds for each  $x_{ij}$  and  $y_{ij}$ . For a direct proof, consider the case

$$\frac{\partial Q}{\partial y_{6i}} \equiv \sum_{i=1}^5 (x_i y_6 x_{i7} + x_i y_7 x_{6i}) + \sum x_{6\mu} x_{\nu 7} y_{\mu\nu} + \sum x_i y_{\nu\rho} y_{\sigma\tau},$$

where in the second sum  $\mu, \nu = 1, \dots, 5; \mu \neq \nu$ ; in the third sum  $i = 1, \dots, 5$ , while  $\nu, \rho, \sigma, \tau$  form a cyclic permutation of 1, 2, 3, 4, 5, with  $i$  excluded. To identify this expression of 45 terms with  $C$ , we may take

$$x_i = z_{6i}, x_{i7} = y_i, x_{6i} = x_i, y_6 = x_6, y_7 = -y_6, y_{ij} = z_{ij} \\ (i, j = 1, \dots, 5; j \neq i).$$

§ 4. Substitution Group  $H$  with the Invariant  $Q$ .

Among the literal substitutions on the 56 letters which leave  $Q$  invariant occur the following types :

$$A : (x_j y_i)(x_i y_{ij}) \quad [i, j = 1, \dots, 7; j \neq i];$$

$$B_{ij} : (x_i y_j)(x_j y_i)(x_{ik} y_{kj})(x_{jk} y_{ki})(x_{ij} y_{ij})(x_{ik} y_{ki})(x_{ik} y_{ki}) \\ [k, l = 1, \dots, 7, \text{excluding } i, j];$$

$$D_1 : (x_i y_1)(x_1 x_{i1})(y_i y_{i1})(x_{jk} y_{jk}) \quad [i, j, k = 2, \dots, 6].$$

Each is the product of 28 transpositions. A suitable product of  $A, B_{ij}, D_1$  throws  $x_i$  to any one of the 56 letters.

The group  $H$  leaving  $Q$  invariant is, therefore, transitive.

The order  $\Omega$  of  $H$  is therefore  $56 \Omega_1$ , where  $\Omega_1$  is the order of the subgroup  $H_1$  which leaves  $x_1$  fixed. The substitutions of  $H_1$  permute amongst themselves the 45 terms of  $\frac{\partial Q}{\partial x_1}$ . In

view of § 2,  $\Omega_1$  is at most equal to  $51840 \Omega'$ , where  $\Omega'$  is the order of the subgroup of  $H$  which leaves fixed the 28 variables  $x_1, y_j, x_{1j}, y_{jk}$  ( $j, k = 2, \dots, 7; j \neq k$ ). We readily verify that the identity is the only substitution  $S$  of  $H$  which leaves these 28 variables fixed, so that  $\Omega' = 1$  and therefore

$$(3) \quad \Omega \equiv 56 \times 51840.$$

In fact,  $S$  must permute amongst themselves the terms of  $Q$  which involve both  $y_j$  and  $x_{1j}$ , and hence leave invariant

$$\frac{\partial^2 Q}{\partial y_j \partial x_{1j}} \equiv x_1 y_{1j} + x_j y_{17} + x_{\nu\rho} x_{\sigma\tau} + x_{\nu\sigma} x_{\tau\rho} + x_{\nu\tau} x_{\rho\sigma},$$

where  $j, \nu, \rho, \sigma, \tau$  is an even permutation of 2, 3, 4, 5, 6. Hence the letters common to two such quadratic forms must be permuted amongst themselves. It follows that  $S$  leaves invariant

$$y_{17}, x_{46}, x_{66}, x_{45}, x_{24}, x_{25}, x_{26}, x_{23}, x_{34}, x_{35}, x_{36}$$

and therefore also  $x_2, x_3, x_4, x_5, x_6$ . Hence each term  $x y_{1j} y_{\nu\rho} y_{\sigma\tau}$  is invariant and therefore  $x_j, y_{1j}$  ( $j = 2, \dots, 7$ ). Similarly,  $x_2 y_{1x_{23}y_{13}}$  and  $x_j y_{1x_{jk}y_{1k}}$  ( $k = 2, \dots, 6$ ) require that  $y_1, x_{1k}$  be fixed. Hence  $S$  leaves all 56 letters fixed and is the identity.

A shorter proof follows from the important lemma

*If a substitution of  $H$  leaves  $x_i$  fixed, it leaves  $y_i$  fixed, and inversely. If it leaves  $x_{ij}$  fixed, it leaves  $y_{ij}$  fixed and, inversely.*

In view of the form of the substitutions  $A, B_{ij}, D_1$  and the transitivity of  $H$ , it suffices to consider the case of a substitution  $\Sigma$  of  $H$  which leaves  $x_1$  fixed. Then  $\Sigma$  permutes the 27 variables  $y_j, x_{1j}, y_{jk}$  ( $j, k = 2, \dots, 7$ ) amongst themselves and therefore permutes the remaining 28 variables  $y_1$  and

$$(4) \quad x_j, y_{1j}, x_{jk} \quad (j, k = 2, \dots, 7; j \neq k).$$

Hence  $\Sigma$  permutes the 45 terms of  $Q$  which involve  $y_1$ , these being the only terms containing exclusively letters of the set (4). Since each of the letters (4) occurs in terms of  $Q$  which do not involve  $y_1$ , no one of the letters (4) is replaced by  $y_1$  under  $\Sigma$ . Hence  $y_1$  is not altered by  $\Sigma$ .

In view of § 5, the group  $H_1$  contains a subgroup holocentrically isomorphic with the group  $G_{51840}^{27}$  of the equation for the 27 lines on a cubic surface. It follows that

$$\Omega \equiv 56 \times 51840.$$

Combining this result with the result (3), we conclude that the order of group  $H$  is  $56 \times 51840$ .

§ 5. *Generators of the Group  $H_7$ .*

In investigating the subgroup  $H_7$  of  $H$  which leaves  $x_7$  fixed, we consider certain substitutions of  $H_7$  which leave  $\frac{\partial Q}{\partial x_7}$  fixed. By § 3, the latter function is identical with the cubic form  $C$  of § 1 if we let  $x_{7k} = -x_k, y_{7k} = z_{jk}$ , for  $k, j = 1, \dots, 6; k \neq j$ . We obtain from the substitutions  $W, E_i, T$  of § 2 the following substitutions leaving  $\frac{\partial Q}{\partial x_7}$  invariant :

$$\begin{aligned}
 [E_2] : & \quad (y_5y_6)(y_4y_{13})(y_3y_{14})(y_1y_{34})(x_{27}y_{56})(x_{67}y_{25}) \\
 & \quad (x_{67}y_{26})(y_{15}y_{16})(y_{35}y_{36})(y_{45}y_{46}), \\
 [E_3] : & \quad (y_2y_6)(y_4y_{13})(y_3y_{14})(y_1y_{34})(x_{27}y_{25})(x_{67}y_{26}) \\
 & \quad (x_{67}y_{56})(y_{12}y_{16})(y_{23}y_{36})(y_{24}y_{46}), \\
 [T] : & \quad (y_4y_6)(x_{47}x_{67})(y_{14}y_{16})(y_{24}y_{26})(y_{34}y_{36})(y_{54}y_{56}),
 \end{aligned}$$

together with  $[W]$  and  $[E_1]$ , whose forms are equally evident. These substitutions do not leave  $Q$  invariant; but leave invariant  $\frac{\partial Q}{\partial x_7}$  and  $\frac{\partial Q}{\partial y_7}$ . By interchanging  $x_i$  with  $y_i$  and  $x_j$  with  $y_j$  ( $i, j = 1, \dots, 7$ ), we obtain the substitutions

$$A^{-1}[W]A, A^{-1}[E_i]A, A^{-1}[T]A \quad (i = 1, 2, 3)$$

affecting only the 27 variables  $x_j, y_{7j}, x_{ij}$  ( $i, j = 1, \dots, 6$ ). Hence they are commutative with  $[W], [E_i], [T]$ . The products

$$(5) \quad [W]A^{-1}[W]A, [E_i]A^{-1}[E_i]A, \quad (i = 1, 2, 3) \\ [T]A^{-1}[T]A$$

therefore generate a group  $K$  which is holodically isomorphic with the group  $G_{61340}^{27}$  of the configuration defined by  $C$ , and hence with the group of the equation for the 27 lines on a cubic surface. Without proof,\* I will state the theorem that the substitutions (5) leave invariant the function  $Q$ .

It follows from § 4 that the group  $K$  generated by the substitutions (5) is identical with the subgroup  $H_7$  of the substitutions of  $H$  which leave  $x_7$  invariant. Hence every substitution  $S$  of  $H_7$  can be expressed as a product  $S_1A^{-1}S_1A$ , where  $S_1$  affects only  $x_{7j}, y_j, y_{7k}$  ( $j = 1, \dots, 6$ ), whereas  $A^{-1}S_1A$  affects only  $y_{7j}, x_j, x_{7k}$ , being similar to  $S_1$ . To give a direct

\* I first verified by direct computation that the theorem is true for  $[E_3]A^{-1}[E_3]A$  and that  $[E_2]A^{-1}[E_2]A$  is the transform of  $[E_3]A^{-1}[E_3]A$  by  $B_{32}$ .

proof, let  $S$  replace  $y_j$  by  $y_{\lambda\mu}$ , for example. There exists a substitution  $R = R_1 A^{-1} R_1 A$ , where  $R_1$  leaves  $x_7$  fixed and affects the same variables as does  $S_1$ , which replaces  $y_j$  by  $y_{\lambda\mu}$  and therefore  $x_j$  by  $x_{\lambda\mu}$ . Then  $SR^{-1}$  leaves  $y_j$  fixed and therefore also  $x_j$ . Hence  $S$  replaces  $x_j$  by  $x_{\lambda\mu}$ .

§ 6. *Isomorphic Substitution Group on 28 Letters.*

THEOREM.—*The group  $H$  of all substitutions on the 56 variables (2) which leave  $Q$  invariant is imprimitive, possessing the 28 systems of imprimitivity*

$$(6) \quad S_i \equiv [x_i, y_i], \quad S_{ij} \equiv [x_{ij}, y_{ij}] \quad (i, j = 1, \dots, 7; j \neq i).$$

Since  $A$  transforms each  $B_{ij}$  and  $D_1$  into themselves, they preserve the imprimitive systems (6). We find that  $A$  corresponds to the identical substitution on  $S_i, S_{ij}$ ; while

$$\begin{aligned} B_{ij} &\sim (S_i S_j)(S_{ik} S_{kj}) & (k = 1, \dots, 7; k \neq i, j), \\ D_1 &\sim (S_k S_{k1}) & (k = 2, \dots, 7), \end{aligned}$$

each substitution on the systems being composed of 6 cycles. A suitable product of them replaces  $S_1$  by any given one of the 28 systems. To the substitutions (5) correspond substitutions on the systems analogous to  $[W]$ ,  $[E_i]$ ,  $[T]$ , respectively. For example, to  $[E_2]A^{-1}[E_2]A$  corresponds the substitution

$$\begin{aligned} (S_5 S_6)(S_4 S_{13})(S_3 S_{14})(S_1 S_{34})(S_{27} S_{56})(S_{57} S_{25}) \\ (S_{67} S_{26})(S_{15} S_{16})(S_{35} S_{36})(S_{45} S_{46}). \end{aligned}$$

*The group  $H$  is hemiedrically isomorphic with a transitive substitution group  $H'$  of order  $28 \times 51840$  on the 28 letters  $S_i, S_{ij}$ .*

There is a known transitive substitution group of the same order and degree, viz., the group  $\Gamma$  of the equation for the 28 bitangents to a quartic curve without double points. By the adjunction of a root,\* the group reduces to a group holoedrically isomorphic with the group  $G$  of the 27 lines on a cubic surface (§ 2). By an earlier result, the subgroup of  $H'$  which leaves one letter fixed is holoedrically isomorphic with  $G$ . It would seem quite probable that  $\Gamma$  and  $H'$  are isomorphic, and indeed identical groups. A formal investigation has not been attempted by the author. Granting the truth of this conjecture, the invariant  $Q$  would define the configuration of the 56 points of contact of the 28 bitangents to a quartic curve without double points.

THE UNIVERSITY OF CHICAGO,  
June, 1901.

\* Jordan, *Traité des substitutions*, p. 330.