may be followed readily by all whom this paper may interest.

Geometrography is treated didactically in the Traité de géométrie of Rouché et de Comberousse (7th edition, volume 1, Gauthier-Villars, Paris, 1900), in the *Archiv der Mathematik und Physik*, April and May, 1901, and more fully in my La géométrografie, Paris, Naud, in press, 8vo. 100 pp.

CONCERNING THE ELLIPTIC $\wp(g_2, g_3, z)$-FUNCTIONS AS COÖRDINATES IN A LINE COMPLEX, AND CERTAIN RELATED THEOREMS.

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Introduction.

SYSTEMS, that have appeared from time to time, of coördinates for the Kummer surface, each more or less related to the elliptic functions, suggest that the existence of such systems of coördinates may be but the partial manifestation of a more general truth; that is to say, since the Kummer surface is definitely related to a line complex of the second order, *i. e.*, is its surface of singularities, any system of coördinates on such a surface ought to arrange itself under a more general system relating at least to the complex of second order, and presumably to the general complex.

The following paper concerns itself with this general question and its application to the Kummer surface and certain other configurations.

§ I.

If we write the general quartic which enters into the discussion of the elliptic functions in the form

$$F(z) \equiv z^4 + \alpha z^3 + \beta z^2 + \gamma z + \delta \equiv \prod_{1,2,3,4.} (z_2^{(\kappa)} z_1 - z_1^{(\kappa)} z_2) = 0,$$

and if

$$(i, \kappa) \equiv \begin{vmatrix} Z_1^{(i)} & Z_2^{(i)} \\ Z_1^{(\kappa)} & Z_2^{(\kappa)} \end{vmatrix},$$

then $F(z)$ has the following irrational invariants:

$$(1, 2)\,(3, 4) \equiv R, \quad (1, 3)\,(4, 2) \equiv S, \quad (1, 4)\,(2, 3) \equiv T,$$

where $R + S + T = 0$.

Put $$A \equiv \frac{R}{6} - \frac{S}{6}, \quad B \equiv \frac{S}{6} - \frac{T}{6}, \quad C \equiv \frac{T}{6} - \frac{R}{6},$$

where again $A + B + C = 0$;

write $$A \equiv \left(\sqrt{\frac{R}{6}} + \sqrt{\frac{S}{6}}\right)\left(\sqrt{\frac{R}{6}} - \sqrt{\frac{S}{6}}\right) \equiv n_1 n_4,$$

$$B \equiv \left(\sqrt{\frac{S}{6}} + \sqrt{\frac{T}{6}}\right)\left(\sqrt{\frac{S}{6}} - \sqrt{\frac{T}{6}}\right) \equiv n_2 n_5,$$

$$C \equiv \left(\sqrt{\frac{T}{6}} + \sqrt{\frac{R}{6}}\right)\left(\sqrt{\frac{T}{6}} - \sqrt{\frac{R}{6}}\right) \equiv n_3 n_6;$$

then $n_1 n_4 + n_2 n_5 + n_3 n_6 = 0$, or say

$$\sum_{1,2,3} n_\lambda n_{\lambda+3} = 0.$$

Then, since $-g_2 \equiv AB + AC + BC$ and $2g_3 \equiv ABC$,

$$-g_2 \equiv n_1 n_2 n_4 n_5 + n_1 n_3 n_4 n_6 + n_2 n_3 n_5 n_6,$$

$$2\, g_3 \equiv n_1 n_2 n_3 n_4 n_5 n_6.$$

§ II.

We may now consider the n_i's either as (a) line-coördinates themselves or (b) as Klein's fundamental complexes.

(a) We put $n_i \equiv p_i$.

Then we have in the first place the two complexes

(1) $$g_2 = -\prod^{\kappa=1,\dots,6}\left\{p_\kappa \sum^{\lambda=1,2,3} \frac{1}{p_\lambda p_{\lambda+3}}\right\},$$

(2) $$g_3 = \tfrac{1}{2}\prod^{1,\dots,6} p_\kappa;$$

write also (3) $x = p_5$, (4) $y = p_6$,

where $z \equiv x + iy$ and hence $z \equiv p_5 + ip_6$.

Together with these four relations we have the identical relation

(5) $$I \equiv p_1 p_4 + p_2 p_5 + p_3 p_6 = 0.$$

Suppose then that we have any complex given by an equation

(6) $$\Omega_n(p_1, p_2, \cdots, p_6) = 0.$$

Consider any definite line in the complex whose coördinates are $P_i \equiv a_i$ $(i = 1, \cdots. 6)$. Substituting these values in equations (1), ⋯, (4), we have

$$g_2 = -\prod^{1,\ldots 6}\left\{a_\kappa \sum^{1,2,3} \frac{1}{a_\lambda a_{\lambda+3}}\right\},$$

$$g_3 = \tfrac{1}{2}\prod^{1\ldots 6} a_\kappa, \quad z = a_5 + ia_6,$$

that is, definite numerical values for g_2, g_3 and z, and hence a definite numerical value for

$$\wp(g_2, g_3, z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^{\,2}}{1200}z^6 + \cdots,$$

i. e., to that line corresponds

(7) $$\wp\left[-\prod^{1\ldots 6}\left\{a_\kappa \sum^{1,2,3}\frac{1}{a_\lambda a_{\lambda+3}}\right\}, \tfrac{1}{2}\prod^{1\ldots 6} a_\lambda,\ a_5 + ia_6\right].$$

Conversely, given a definite $\wp$-function, from it we can determine g_2, g_3, and z, numerically, say

$$g_2 = h_1, \quad g_3 = h_2, \quad z = h_3 + ih_4.$$

Substituting these values in equations (1), ⋯, (5), we have

(8) $$-\prod^{1\cdots 6}\left\{p_\kappa \sum^{1\cdots 3}\frac{1}{p_\lambda p_{\lambda+3}}\right\} = h_1,$$

(9) $$\tfrac{1}{2}\prod^{1\cdots 6} p_\kappa = h_2,$$

(10) $$p_5 = h_3,$$ (11) $$p_6 = h_4,$$

(12) $$p_1 p_4 + p_2 p_5 + p_3 p_6 = 0.$$

These together with the equation of the complex

(6) $$\Omega_n(p_1, \cdots, p_6) = 0$$

make six equations from which to determine the six quantities p_κ ($\kappa = 1, \cdots, 6$) in terms of h_i and the coefficients of (6).

If $p_6 = 0$ then no other p_i can vanish, so that the *$\wp$-function is not infinite for any real line of the complex.* If we put $p_6 = ip_5$ then the $\wp$-function is infinite and we may say that the *infinite values of the $\wp$-function correspond to a definite set of imaginary lines.*

For lines of the complex with finite coördinates we have $p_6 \neq 0$. Then if any other p_κ ($\kappa = 1, \cdots, 5$) vanish we have

$$(13)\quad \begin{aligned} &g_3 = 0, \\ &g_2 = -p_s p_{s+3} p_t p_{t+3}, \\ &I \equiv p_s p_{s+3} + p_t p_{t+3} = 0 \end{aligned} \qquad \begin{aligned} &(s,\ t = 1, \cdots, 5\ ;\ s \neq t,\ \kappa\ ; \\ &\ t \neq \kappa,\ s + 3). \end{aligned}$$

From these we find

$$p_t p_{t+3} = \sqrt{g_2}, \quad p_s p_{s+3} = -\sqrt{g_2},$$

so that the equation of the complex takes the form

$$(14)\qquad F\left(\frac{\sqrt{g_2}}{p_t},\ -\frac{\sqrt{g_2}}{p_s},\ p_s,\ p_s,\ p_s\right) = 0,$$

where p_r is the coefficient of p_κ in the identity. Hence:

The vanishing of g_3 characterizes the ruled surface

$$F\left(\frac{\sqrt{g_2}}{p_t}, -\frac{\sqrt{g_2}}{p_s}, p_t, p_s, p_r\right) = 0.$$

The corresponding $\wp$-function is $\wp\,(p_t^2 p_{t+3}^2,\ 0,\ p_5 + ip_6)$.

If g_2 vanish, we have

$$p_1 p_2 p_4 p_5 = -(p_1 p_3 p_4 p_6 + p_2 p_3 p_5 p_6).$$

From (2) the left member is equal to $\dfrac{2g_3}{p_4 p_6}$, which gives

$$(p_3 p_6)^2 = \frac{-2g_3}{p_1 p_4 + p_2 p_5},$$

or, making use of the identical relation $I = 0$,

$$p_3 p_6 = \sqrt[3]{2g_3}\ ;$$

with this value we find

$$p_1p_4 = \omega^2\sqrt[3]{2g_3}, \quad p_2p_5 = \omega\sqrt[3]{2g_3},$$

where ω is an imaginary cube root of unity, so that for $g_2 = 0$ the identical relation is simply

$$\omega^2 + \omega + 1 = 0.$$

The equation of the complex becomes then, *when* $g_2 = 0$,

$$\Omega_n\left[\omega^2\frac{\sqrt[3]{2g_3}}{p_4},\ \omega\frac{\sqrt[3]{2g_3}}{p_5},\ \frac{\sqrt[3]{2g_3}}{p_6},\ p_4,\ p_5,\ p_6\right] = 0$$

which is a ruled surface; the identical relation becomes

$$\omega^2 + \omega + 1 = 0.$$

The corresponding $\wp$*-function is*

$$\wp\{0, \tfrac{1}{2}p_1^3p_4^3, p_5 + ip_6.\}$$

§ III.

Consider next Klein's fundamental complexes and put

$$A = x_1^2 + x_4^2; \quad B = x_2^2 + x_5^2; \quad C = x_3^2 + x_6^2,$$

so that

$$\sum^{1\ldots6} x_i^2 = 0.$$

Then we have

15 $$g_3 = \tfrac{1}{2}\prod^{1,2,3}(x_i^2 + x_{i+3}^2), \qquad g_2 = \sum^{1.2.3}\frac{-g_3}{x_i^2 + x_{i+3}^2}$$

and in the value of z we write $x = x_4$ and $y = x_6$, so that $z = x_4 + ix_6$.

Then, as in the first case, we shall have always six equations to determine the x's; these in turn give six equations to determine the p's, or true coördinates.

For a directrix of the congruence determined by the two complexes x_1 and x_2, say the congruence (x_1x_2) we have $x_1^2 + x_2^2 = 0$, that is, an edge of a definite fundamental tetrahedron. Each of the three factors of g_3 corresponds to two, and hence all three factors to the six edges of this tetrahedron. *Hence the tetrahedron formed by the directrices of congruences* (x_1x_2), (x_3x_4), *and* (x_5x_6) *is characterized by* $g_3 = 0$. *The corresponding* $\wp$*-functions are* $\wp(0, 0, \infty)$; $\wp(0, 0, \pm 1)$.

But the particular interest which attaches to these fundamental complexes is their easy application to the complex of second order, viz., the equation then—as Klein shows—is simply $\sum \kappa_i x_i^2 = 0$. Also

$$x_i^2 = \frac{\prod^{p=1,2,3,4} (x_i - \lambda_p)}{\rho f'(x_i)}, \qquad (i = 1,\cdots, 6),$$

where

$$f(\lambda) \equiv \Pi(x_i - \lambda)$$

and the λ_p are roots of

$$\sum_{1\cdots 6} \frac{x_i^2}{x_i - \lambda} = 0.*$$

Then we have

$$x_\alpha^2 + x_\beta^2 = \frac{1}{\rho f'(x_\alpha) f'(x_\beta)} \begin{vmatrix} f'(x_\beta) \Pi(x_\beta - \lambda_p) \\ f'(x_\alpha) \Pi(x_\alpha - \lambda_p) \end{vmatrix}$$

or say

$$\frac{\varDelta_{\alpha\beta}}{\rho f'(x_\alpha) f'(x_\beta)};$$

whence

$$g_3 = \frac{1}{2\rho^3} \prod^{1,2,3} \frac{\varDelta_{i,\,i+3}}{f'(x_i) f'(x_{i+3})}, \quad g_2 = \rho \Sigma \frac{-g_3 f'(x_i) f'(x_{i+3})}{\varDelta_{i,\,i+3}}.$$

For the tangent to the Kummer surface two λ's are equal and for the 16 double planes or double points all four are equal.

Write β as the value of the equal λ's. Also put

$$P_{s,t}(m, n, r) \equiv \begin{vmatrix} (x_s - \beta)^m i & \left[\dfrac{[x_t - \lambda_1 \cdot x_t - \lambda_2]^m}{\rho f'(x_t)} \right]^r \\ (x_t - \beta)^m & \left[\dfrac{[x_s - \lambda_1 \cdot x_s - \lambda_2]^n}{\rho f'(x_s)} \right]^r \end{vmatrix}$$

Then the Kummer surface is characterized by

$$\wp \left\{ \Sigma \frac{\Pi P_{s,\,s+3}(2,1,1)}{P_{i,\,i+3}(2,1,1)}, \Pi P_{s,\,s+3}(2,1,1), P_{4,6}(1,1,\tfrac{1}{2}) \right\}$$

and its 16 double planes and double points by

$$\wp \left\{ \Sigma \frac{\Pi P_{s,\,s+3}(401)}{P_{i,\,i+3}(401)}, \Pi P_{s,\,s+3}(401), P_{4,6}(2,\ 0,\ \tfrac{1}{2}) \right\}$$

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* Klein, *Math. Annalen*, vol. 5, pp. 294–5.