EUCLID'S ELEMENTS.

Euclid und die sechs planimetrischen Bücher. Mit Benutzung der Textausgabe von Heiberg. Von Dr. Max Simon. Leipzig, Teubner, 1901. 8vo. vi + 141 pp.

The Contents of the Fifth and Sixth Books of Euclid. Arranged and explained by M. J. M. Hill, F.R.S. Cambridge University Press, 1900. 4to. xix + 243 pp.

It is probable that, whether we are glad of it or not, Euclid has been banished from our American schools, never to return. Yet there is a real need for better professional knowledge among our teachers of geometry, and for this reason we welcome Dr. Simon's book, which fairly bristles with remarks helpful even to those teachers who rely upon the most iconoclastic of our text-books.

The book begins with a short account of Euclid's writings, and an elaborate bibliography of the "Elements." The choice of books to which reference is made, seems a little arbitrary. For instance, the author mentions (page 14) the fifteen editions of Legendre, published between 1794 and 1852, under the head of "Efforts to supersede Euclid," but passes over in silence such important work as the "Syllabus" of the "Association for the Improvement of Geometrical Teaching," not to speak of any more recent literature. In fact, he seems to have cared little for English sources, except Simson; for, though he mentions Hoüel's defence of Euclid against Legendre, he has no word for that most truly charming bit of textbook criticism, Dodgson's "Euclid and his modern rivals."

The introduction is followed by the twenty-two definitions of book I, with a note to each. In referring to the complaint frequently urged that some of Euclid's definitions would be no help to any one who had previously no idea of the objects in question, the author quotes from Lambert to the effect (page 25) that Euclid was doubtless aware of this, and in giving the definitions he was merely acting as the artisan who shows his apprentice around the shop, telling him the names of the various implements. The remarks upon the definitions are for the most part good, especially those which deal with the "point." In defining the angle, Dr. Simon uses the word "Biegung" instead of the more common "Neigung," and defends the innovation at some length We are not convinced, however, that this is a change for the better, for an idea of curvature or bending seems to lurk about "Biegung" which is certainly not wanted in the definition of a plane angle. The notes upon the axioms are less satisfactory. The author leaves us in doubt about his own opinion as to the possibility of proving that all right angles are equal, merely showing that the proof of Geminos is inadequate.

The criticisms of the propositions in the first book are perhaps the most instructive of any. For instance, he points out how the proof of the first congruence theorem for triangles depends upon the axiom of free mobility, or the homaloidal nature of space. It is a pity that he does not dwell at even greater length upon this point, for Hilbert* and Veronese† assume this theorem as an axiom. Yet something may still be said upon the side of those who look upon geometrical congruence as meaningless, except by means of possible superposition, and hence regard the introduction of such an axiom as superfluous. ‡

On page 51 is the proof of the proposition that an exterior angle of a triangle is greater than either of the opposite interior angles. The author remarks that this was used by Legendre to show that the sum of the angles of a triangle cannot be greater than two right angles; but many years before, Saccheri turned it to a similar use. Dr. Simon neglects to show how this proof may break down when a straight line is closed.

The notes on books two and three are less copious, with the interesting exception of pages 87–90, where we have an account of the dispute over the magnitude of the angle between two tangent circles. At the beginning of book five are some particularly instructive notes upon the intimate connection between the euclidean theory of proportion, and our modern idea of irrational numbers. The author goes so far as to say (page 109), "It appears, in my opinion, indisputably, from the fifth book that they (the Greeks) possessed the idea of number with complete sharpness, nearly identical with Weierstrass's conception of it." On the next page occurs a singular omission, where the author totally fails to grasp the significance of Archimedes's axiom. He merely calls it a postulate to the effect that lengths may be compared as to magnitude, and used in arithmetical processes. But the axiom of Archimedes is primarily one of continuity, and what is more it is possible to construct non-

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* Hilbert, Die Grundlagen der Geometrie, p. 12.

† Veronese, Grundzüge der Geometrie von mehreren Dimensionen, p. 260.

‡ Cf Russell, The Foundations of Geometry, pp. 147–158.

archimedean number systems, and geometries.* True, it is hard to imagine just what such geometries are like. We are in the habit of assuming † that if a variable increase unceasingly it will either approach a limit or eventually surpass any chosen value ; but if in non-archimedean geometry we lay off the same length an endless number of times on a straight line, neither of these will occur, the latter being prohibited by definition, while the former is at variance with our definitions of fixed length and limit.

One more omission is worth speaking of : it occurs in the note to the eighteenth proposition in book six. This is a theorem to show that upon any segment of a straight line, we may construct a figure similar to a given figure, and similarly placed. The construction consists, of course, of a composition of similar triangles ; but a rigorous proof should be explicitly based on the method of mathematical induction, which is tacitly used, and also show that the construction is unique. Euclid naturally goes into neither of these subtleties, but their absence should certainly be noticed in a careful commentary.

In conclusion, it is worth mentioning that Dr. Simon's commentaries are all of a historical, philological, or mathematical nature, and untrammeled by pearls of pedagogical wisdom. This is a wise limitation, for an equally adequate treatment of such debatable matter as methods of teaching Euclid would have changed the book from a compact to a voluminous one, besides being foreign to the purpose of the historical series of which it forms a part.‡ Yet it is a very human work withal, as may be seen from the following anecdote, which the writer narrates (page 19). "I found a construction for a tangent to the 'limiting circle' which seemed to me equally simple and new for our old circle. I announced it in Munich where were assembled one hundred mathematicians, nearly all teachers of distinction in the universities ; and neither I nor any one else had a suspicion that it was Euclid's construction." (Book 3, proposition 17.)

The fact, mentioned above, that Euclid no longer presents a living issue in American education makes it difficult to pass a fair judgment on Dr. Hill's book. We are inclined to look upon attempts to "temper the wind to the shorn

* Hilbert, l. c. cit., p. 25 ; Dehn, "Die Legendre'schen Sätze über die Winkelsumme im Dreieck," *Math. Annalen*, vol. 53.

† Conf. Osgood, Introduction to Infinite Series, p 14.

‡ Abhandlungen zur Geschichte der mathematischen Wissenschaften.

lamb" by elucidating particular spots in the "Elements" as a waste of time and energy. But we have no right to start from such an assumption in considering the present work. The simple fact is that Euclid has been, and will long continue to be, the foundation of geometrical knowledge in that nation which has produced Newton and Cayley and Sylvester. Parts of Euclid are undoubtedly too difficult for beginners, and the book before us attempts to remove the greatest of these difficulties, the theory of proportion. In American books we seek to reach this end by an appeal to the analogy of algebra, but herein we depart entirely from Euclid's pattern. Professor Hill adopts a different, and more euclidean device: the relative multiple scale.

Let A and B be two magnitudes which are supposed to be capable of comparison as well as their multiples. The order of succession of the various multiples of A and B is called their relative multiple scale, and may be represented symbolically by $rA \lesseqqgtr sB$ where r and s are positive integers. The relative multiple scales of A,B and C,D are equal if, for every positive integral value of r and s, $rA \gtreqqless sB$ is a sufficient condition for $rC \gtreqqless sD$. A simple geometrical diagram is introduced to exhibit the three possible relations and this diagram is continually presented to give the learner a clear idea of what he is doing. Two sets of magnitudes having equal multiple scales are said to be proportional, and the transformations which leave the proportional relation unaltered constitute a large part of the book. With perfect propriety, the author lays great stress upon the axiom of Archimedes, and there is throughout a commendable consistency in the method and spirit of the work.

Such is the general scheme and, granted the author's aim, and point of view, it is about the best that could be adopted. In some of the details, the book is almost incredibly careless.

The first proposition (page 2) is a proof of the commutative and associative laws in the multiplication of positive integers, but the proofs are based upon the assumption that these laws hold in the case of addition. On page 6 is the proof that if a and b are positive integers, while $a > b$, and R is a magnitude $aR > bR$. It is tacitly assumed that $bR + cR > bR$, or that R obeys axiom 9, "The whole is greater than its part."

A multiple scale is defined (page 12) as a set of magnitudes, while a relative multiple scale is a portion of a diagram (page 14). On page 29 is the corollary "If ABC be a triangle, and if the sides AB, AC be cut by any straight line parallel to BC, then the sides AB, AC are divided proportionally." This is given before the word "proportionally" is defined.

Section 3, pages 33–36, is entitled "A Chapter on Ratio," and from it we glean the following facts: The reader is supposed to have an innate idea of relative magnitude. "The ratio of one magnitude to another (which must be of the same kind as the first) is the relative magnitude of the first compared with the second." Having now a clear and precise idea of what a ratio is, from this soul-satisfying definition, we are prepared for the next step, which is this: If we accept the axiom that a number ρ may always be found of such a sort that the relative multiple scale of ρ and 1 is equal to that of the two comparable magnitudes A and B, then ρ may be taken as the measure of the ratio of any pair of magnitudes having the same scale as A and B. It is interesting to notice that the author himself seems to have a certain diffidence about these definitions, for he does not use them in any of his subsequent work, but sticks to the safe path of the relative multiple scale. Apparently their only use in the whole chapter is that it is convenient to use the words "ratio" and "proportion" while Euclid's definition of the latter is cumbersome.

As a last error in detail, we notice the following definition on page 73:

"If four straight lines (presumably in the same plane) be cut by any transverse in four harmonic points, they are called four harmonic lines, or said to form a harmonic pencil." This may be interpreted in two ways. If the author means that they either form four harmonic lines or a harmonic pencil, the former seems a perfectly useless definition of four harmonic tangents to a conic; if, on the other hand, he looks upon the two forms of words as synonymous, and this seems the more natural interpretation, he is departing from the usual English practice of defining a pencil as composed of coplanar and concurrent lines.*

To sum up: the book seems to us well planned but carelessly executed. It has, of course, no message for the average American teacher, and whoever uses it must do so with circumspection. With this caution it may prove to be of some value to teachers who do not care to depart entirely from Euclid's theory of proportion, yet find the ordinary presentation beyond the grasp of their pupils.

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*Conf. Cremona, Projective Geometry, p. 22.