THE APPLICATION OF THE FUNDAMENTAL LAWS OF ALGEBRA TO THE MULTIPLICATION OF INFINITE SERIES.

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THE present writer has given examples in which an absolutely convergent series is obtained as the result of multiplying two conditionally convergent series together, or of multiplying one conditionally convergent series by a divergent series.*

He has also given an example of two divergent series whose product is absolutely convergent.†

Pringsheim has treated this subject from a more general point of view and by very simple methods has shown that the property in question is typical of certain classes of series.‡

In the present paper it is proposed to establish a class of series with real terms, possessing the property alluded to, but which seems to be distinct from the class given by Pringsheim. Next, we shall consider the validity of the fundamental laws of algebra in the multiplication of infinite series. Then, with aid of our conclusions relating to these laws, we shall point out another method for obtaining divergent series whose product is absolutely convergent. Lastly we shall generalize a theorem of Abel on the multiplication of series.

§ 1.

In the series S_1 and S_2, obtained respectively by removing the parentheses from the series

$$S_1' \equiv \sum_{v=0}^{v=\infty} (a_{4v} - a_{4v+1} + a_{4v+2} - a_{4v+3}),$$

$$S_2' \equiv \sum_{v=0}^{v=\infty} (b_{4v} + b_{4v+1} - b_{4v+2} - b_{4v+3}),$$

wherein the a's and b's are real and positive, let the following conditions be satisfied :

(1) The vth term in S_1 and in S_2 shall be $\gtreqq v^{-r}$, where $\frac{1}{2} < r \gtreqq 1$, but $\sum a_s$ and $\sum b_s$ are both divergent.

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* *Trans. Amer. Math. Society*, vol. 2, pp. 25–36, January, 1901.

† *Science*, new ser., vol. 14, p. 395 (September 13, 1901)

‡ *Trans. Amer. Math. Society*, vol. 2, pp. 404–412, October, 1901.

(2) $a_{4v} = a_{4v+2},\ a_{4v+1} = a_{4v+3},\ b_{4v} = b_{4v+1},\ b_{4v+2} = b_{4v+3}$;

the binomials

$$a_{4v} - a_{4v+1}, \quad b_{4v+2} - b_{4v},$$

$$a_{4(v+1)} - a_{4v+1}, \quad b_{4v+2} - b_{4(v+1)},$$

shall each differ from ε_{4v+2} by less than k_{4v+2}, where

$$\varepsilon_s = \frac{1}{s(lg\, s)^\lambda}, \quad \lambda > 1, \quad \text{and} \quad k_s = \frac{1}{s^{2-r}(lg\, s)^\lambda}.$$

The $(4n)$th term of the product of S_1 and S_2 is

$$\sum_{v=0}^{v=n-1} (- b_{4v} \cdot a_{4(n-v-1)+3} + b_{4v+1} \cdot a_{4(n-v-1)+2} + b_{4v+2} \cdot a_{4(n-v-1)+1} - b_{4v+3} \cdot a_{4(n-v-1)}),$$

$$= \sum_{v=0}^{v=n'-1} \{ b_{4v}(a_{4(n-v-1)+2} - a_{4(n-v-1)+3}) - b_{4v+2}(a_{4(n-v-1)} - a_{4(n-v-1)+1}) \}$$

$$+ \sum_{v=n'}^{v=n-1} \{ a_{4(n-v-1)+1}(b_{4v+2} - b_{4v}) - a_{4(n-v-1)}(b_{4v+3} - b_{4v+1}) \}$$

$$= \sum_{v=0}^{v=n'-1} (b_{4v} - b_{4v+2})(a_{4(n-v-1)} - a_{4(n-v-1)+1})$$

$$+ \sum_{v=n'}^{v=n-1} (b_{4v} - b_{4v+2})(a_{4(n-v-1)} - a_{4(n-v-1)+1}),$$

or numerically

$$< \sum_{v=0}^{v=n'-1} 4\varepsilon_{4v+2} \cdot \varepsilon_{4(n-v-1)} + \sum_{v=n'}^{v=n-1} 4\varepsilon_{4v+2} \cdot \varepsilon_{4(n-v-1)}$$

$$< c\varepsilon_{4(n-n')} + c'\varepsilon_{4n'}, \tag{I}$$

where

$$c \geqq \sum_{v=0}^{v=n'-1} 4\varepsilon_{4v+2}, \quad c_1' \geqq \sum_{v=n'}^{v=n-1} 4\varepsilon_{4(n-v-1)},$$

$n' = \frac{1}{2}(n-2)$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Each of the terms in (I) is of the same order of magnitude as $\dfrac{1}{4n(lg\, 4n)^\lambda}$.

The same reasoning which we used with the $(4n)$th term can be applied to the $(4n+1)$th, $(4n+2)$th, $(4n+3)$th term. Thus in the product of the two conditionally convergent series S_1 and S_2 each term is numerically less than the corresponding term of a series known to be absolutely convergent. Hence the product of S_1 and S_2 is absolutely convergent.

As an example, we give the two series T_1 and T_2, obtained respectively by dropping the parentheses from the following series:

$$T_1' \equiv \sum_{v=0}^{v=\infty}\left(\frac{1}{4v^r+1}-\frac{1}{4v^r+4}+\frac{1}{4v^r+1}-\frac{1}{4v^r+4}\right),$$

$$T_2' \equiv \sum_{v=0}^{v=\infty}\left(\frac{1}{4v^s+4}+\frac{1}{4v^s+4}-\frac{1}{4v^s+1}-\frac{1}{4v^s+1}\right),$$

where $\frac{1}{2} < r \gtreqless 1$ and $\frac{1}{2} < s \gtreqless 1$. T_1 and T_2 are each conditionally convergent; their product is absolutely convergent.

§ 2.

The behavior of infinite series with respect to the fundamental laws of algebra may be considered under two heads: An inquiry into the validity of the laws (1) when applied to the *terms* of an infinite series, (2) when applied to the infinite series *themselves*.

The first inquiry has led to the result that the associative law can always be applied to the terms of a convergent infinite series, but that the commutative law can be applied, in general, only to the terms of an absolutely convergent series.

The second inquiry has been made for the addition (and subtraction) of infinite series but, so far as we have seen, not for their multiplication with each other.

The product of

$$U \equiv \sum_{n=0}^{n=\infty} u_n \quad \text{and} \quad V \equiv \sum_{n=0}^{n=\infty} v_n$$

has been defined by Cauchy to be

$$\sum_{n=0}^{n=\infty}(u_0v_n + u_1v_{n-1} + \cdots + u_nv_0).$$

Law of Association.—This law can be applied without limitation to the multiplication of series.

To show this, let $W \equiv \sum\limits_{n=0}^{n=\infty} w_n$, where w_n, as well as u_n and v_n given above, are finite constant numbers, real or complex. Then we have $(UV)W = U(VW)$, for the $(n+1)$th term in the product $(UV)W$ is

$$\begin{aligned}
&(u_0v_n + u_1v_{n-1} + \cdots + u_nv_0)w_0 \\
&+ (u_0v_{n-1} + u_1v_{n-2} + \cdots + u_{n-1}\, v_0)w_1 \\
&+ (u_0v_{n-2} + \cdots + u_{n-2}\, v_0)w_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&+ (u_0v_0)w_n.
\end{aligned}$$

The $(n+1)$th term of the product $U(VW)$ is

$$\begin{aligned}
&u_0\,(w_0v_n + w_1v_{n-1} + \cdots + w_nv_0) \\
&+ u_1\,(w_0v_{n-1} + w_1v_{n-2} + \cdots + w_{n-1}\, v_0) \\
&+ u_2\,(w_0v_{n-2} + \cdots + w_{n-2}\, v_0) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&+ u_n\,(w_0v_0).
\end{aligned}$$

These two expressions for the $(n+1)$th term, for any positive integral value of n, no matter how large, are seen to be identical as soon as we give our assent to the following two statements:

1. For $n > q$, where q is any positive finite number, we have always

$$u_p(w_0v_{n-p} + w_1v_{n-p-1} + \cdots + w_{n-p}v_0) = u_pw_0v_{n-p} + \cdots + u_pw_{n-p}\, v_0.$$

2. For $n > q$, we are allowed to commute the terms obtained by removing the parentheses, provided of course that no terms be dropped from the total aggregate and no new terms be admitted to it. It will be seen that this special case does not contradict our previous statement that the commutative law is not, in general, applicable to the terms of series not absolutely convergent.

The first expression for the $(n+1)$th term assumes the form of the second if in the first we perform in each row the indicated multiplication, then add the columns from left to right, and factor.

Since the $(n+1)$th term in $(UV)\,W$ is the same as the $(n+1)$th term in $U(VW)$, no matter what positive integral value be assigned to n, it follows that the two products are identical. Thus the associative law is always obeyed.

Law of Commutation.—Cauchy's definition makes the product $\sum u_n \cdot \sum v_n$ the same as the product $\sum v_n \cdot \sum u_n$, so that the commutative law holds for two factor series. Being permitted to assume the associative law, it follows easily that the commutative law is valid for three or more factor series. Thus,

$$UVW = U(VW) = U(WV) = UWV = (UV)W = (VU)W = VUW = V(WU) = VWU = W(UV) = WUV = W(VU) = WVU.$$

Law of Distribution.—That $U(V + W) = UV + UW$ can be shown in a manner similar to our proof of the associative law, *viz:* find the $(n + 1)$th term of $U(V + W)$ and the $(n + 1)$th term for $UV + UW$. Then assuming the commutative and distributive laws to hold for the aggregate of terms involved in one of the expressions, change it into the other.

§ 3.

Proceeding as does Pringsheim, let

$$f(x) = \sum_{v=0}^{v=\infty} (-1)^v a_v x^v, \tag{II.}$$

where a_v is of the same order of magnitude as v^{-r}, $\frac{1}{2} < r < 1$. Here $x = -1$ is a singular point on the circle of convergence and $f(-1)$ is an infinity of the same order as $(1 - 1)^{-(1-r)}$. If the series (II) is raised to the positive integral power p, then the sum of the resulting series, for $x = -1$, is of the same order of infinity as $(1-1)^{-p(1-r)}$. If the power $p \gtreqless \frac{1}{1-r}$, then the order of infinity is not lower than the first. But, for $x = +1$, series (II) becomes a special case of S_1 of § 1. Hence the pth power of S_1 is divergent, when a_v is of the same order of magnitude as v^{-r}.*

In the same way it can be shown that the pth power of series S_2 is divergent, when b_v is a magnitude of the same order as v^{-r}, $\frac{1}{2} < r < 1$, and $p \gtreqless \frac{1}{1-r}$. But $S_1 \cdot S_2$ was shown to be absolutely convergent. We have $S_1 \cdot S_2 \cdot S_1 \cdot S_2 \cdots$ (to p pairs of factors) $= (S_1 S_2)(S_1 S_2) \cdots$ (to p parentheses). Hence the product of these $2p$ series is absolutely convergent. But, by the associative and commutative laws, this product is equal to $S_1^p \cdot S_2^p$. Thus, S_1^p and S_2^p are two divergent series whose product is absolutely convergent. Observe that, no matter how much p is in excess of $\frac{1}{1-r}$—that is, no matter how high a power of S_1 and S_2 is taken—we have, for a given value of r, always an absolutely convergent product result-

* Pringsheim, *loc. cit.*, pp. 409–411.

ing from the multiplication of S_1^p by S_2^p. Special example: $T_1^3 \cdot T_2^3$ is absolutely convergent, but T_1^3 and T_2^3 are each divergent when $r < \frac{2}{3}$ and $s < \frac{2}{3}$.

In the divergent series S_1^p the terms increase without limit in numerical value, as v increases without limit. The same is true of S_2^p. Herein lies the difference between this pair of divergent series yielding an absolutely convergent product, and the pair given by Pringsheim.* In the latter the terms of the divergent series remain finite as v increases indefinitely.

From the relation $S_1S_2S_1S_2 \cdots = S_1^p \cdot S_2^p$ we see that there are cases in the multiplication of series in which divergent series may be used with safety—the sum of the final product series being convergent and equal to the product of the sums of the initially given convergent factor series, even when the product of *some* of the given factor series is divergent.

If two or more convergent series, when multiplied together, yield a convergent product series, then the sum of this product series is equal to the product of the sums of the factor series.

This theorem was proved by Abel for the case of two factor series,† and his method of proof is applicable to the general case. The extension is obvious.

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CONCERNING THE CLASS OF A GROUP OF ORDER p^m THAT CONTAINS AN OPERATOR OF ORDER p^{m-2} OR p^{m-3}, p BEING A PRIME.

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IF a non-abelian group of order p^m contains an operator of order p^{m-1} it is of the second class.‡ It is the object of

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* *Loc. cit.*, p. 409.

† *Oeuvres complètes* de N. H. Abel, Tome Premier, 1839, "Recherche sur la série $1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \ldots$," Theorem VI.

‡ Burnside, Theory of Groups, p. 76. If we form the group of cogredient isomorphisms G' of G, then the group of cogredient isomorphisms G'' of G', and so on we finally come either to identity or to a group that has no invariant operators except identity, and is therefore simply isomorphic with its group of cogredient isomorphisms. The groups for which this process leads to identity are classified according to the number of these successive groups of cogredient isomorphisms.