NOTE ON ISOTROPIC CONGRUENCES.

BY DR. L. P. EISENHART.

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CONSIDER a sphere S of radius unity and center at the origin of coördinates, and a surface S_1 corresponding to S by orthogonality of linear elements. By a theorem of Ribaucour * we know that S_1 is the mean surface of an isotropic congruence C. If S_1 is taken to define an infinitesimal deformation of S, the associate† surface in this deformation will be a minimal surface S_2 ;‡ then S and S_2 correspond by parallelism of tangent planes at corresponding points. Moreover, if the reciprocal character of the relation existing between these latter two surfaces is noted, and S is considered as the associate in the deformation of S_2, then the surface S_3 corresponding to S_2 with orthogonality of linear elements is the adjoint minimal surface of S_2.

Darboux has shown § that the following relations exist between the cartesian coördinates of these four surfaces :

$$
\begin{aligned}
x_1 &= x_3 - y_2 z + z_2 y, \\
y_1 &= y_3 - z_2 x + x_2 z, \qquad (1)\\
z_1 &= z_3 - x_2 y + y_2 x.
\end{aligned}
$$

Let S be referred to its asymptotic lines ; then S_2 and S_3 will be referred to the double system of lines which is conjugate for each ‖ As the latter are adjoint minimal surfaces, this double system is made up of the lines of length zero on each surface. Weierstrass has shown ** that the coördinates of S_3 can be expressed as the following functions of u and v, parameters referring to the lines of length zero :

$$
\begin{aligned}
x_3 = \frac{1-u^2}{2} f''(u) + uf'(u) - f(u) \\
+ \frac{1-v^2}{2} f_1''(v) + vf_1'(v) - f_1(v),
\end{aligned}
$$

* Étude des Elassoïdes, chap. 8 (Mémoire couronné par l'Academie de Belgique) ; Darboux, vol. 4, p. 15.

† Bianchi, p. 279.

‡ Darboux, vol. 4, p. 96.

§ Ibid., p. 66.

‖ Bianchi, p. 284.

** Darboux, vol. 1, p. 289.

$$
(2) \qquad \begin{aligned}
y_3 &= i\frac{1+u^2}{2}f''(u) - iuf'(u) + if(u) \\
&\quad - i\frac{1+v^2}{2}f_1''(v) + ivf_1'(v) - if_1(v), \\
z_3 &= uf''(u) - f'(u) + vf_1''(v) - f_1'(v),
\end{aligned}
$$

where accents denote differentiation. Since S_2 is the adjoint minimal surface of S_3, the coördinates x_2, y_2, z_2 have the following expressions :*

$$
(3) \qquad \begin{aligned}
x_2 &= i\left[\left(\frac{1-u^2}{2}f''(u) + uf'(u) - f(u)\right)\right. \\
&\qquad \left. - \left(\frac{1-v^2}{2}f_1''(v) + vf_1'(v) - f_1(v)\right)\right], \\
y_2 &= -\left(\frac{1+u^2}{2}f''(u) - uf'(u) + f(u)\right. \\
&\qquad \left. + \frac{1+v^2}{2}f_1''(v) - vf_1'(v) + f_1(v)\right), \\
z_2 &= i\left(uf''(u) - f'(u) - vf_1''(v) + f_1'(v)\right).
\end{aligned}
$$

Since S and S_2 have parallel tangent planes at corresponding points and S is a sphere of radius unity, the coördinates x, y, z are equal to the direction cosines of the normal to S. Hence these coördinates have the following expressions : †

$$
(4) \qquad x = \frac{u-v}{1+uv}, \quad y = i\frac{v-u}{1+uv}, \quad z = \frac{uv-1}{1+uv}.
$$

Substituting in (1) the above expression for $x, \cdots, z_3$, we find

$$
(5) \qquad \begin{aligned}
x_1 &= \frac{u+v}{1+uv}(f' + f_1') - 2\frac{f+f_1}{1+uv}, \\
y_1 &= i\frac{v-u}{1+uv}(f' + f_1') - 2i\frac{f_1 - f}{1+uv}, \\
z_1 &= \frac{uv-1}{1+uv}(f' + f_1') - 2\frac{uf_1 + vf}{1+uv}.
\end{aligned}
$$

But these are the expressions found by Darboux ‡ for the

* Darboux, vol. 1, p. 323.
† Ibid., p. 296.
‡ Darboux, vol. 4, p. 17.

cartesian coördinates of a point on the mean surface of an isotropic congruence, where the functions f and f_1 which enter in these expressions refer to the minimal surface which is the envelope of the mean planes of the congruence. In the present case these functions refer to S_3, the adjoint minimal surface of S_2, the latter being the associate of S in the deformation determined by S_1. Hence we have the following theorem :

The middle envelope of an isotropic congruence is the adjoint of the minimal surface which is the associate surface in the infinitesimal deformation of the sphere, the directrix of the congruence.

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KRONECKER'S LECTURES ON THE THEORY OF NUMBERS.

Vorlesungen über Mathematik, von Leopold Kronecker, herausgegeben unter Mitwirkung einer von der Königlich Preussischen Akademie der Wissenschaften eingesetzten Kommission. In zwei Teilen. Zweiter Teil. *Vorlesungen über Allgemeine Arithmetik.* Bearbeitet und herausgegeben von DR. KURT HENSEL. Erster Abschnitt : *Vorlesungen über Zahlentheorie.* Erster Band. 8vo., xvi + 509 pp. Teubner, 1901.

IN the summer semester of 1841 Lejeune Dirichlet gave, for the first time, a course of lectures under the title of "Zahlentheorie" at the university of Berlin.* These lectures were attended by Kronecker. The subject was soon afterward added to the regular announcements not only at Berlin but also at the other German universities. The fact that during the winter semester of the current university year at least seven of the German universities offered courses on this subject, given by such well-known men as Frobenius, Weber, and Gordan, is sufficient evidence of the abiding interest in the theory which, according to Gauss, excels all other parts of pure mathematics in "its magic charm" and "its inexhaustible richness."

The volume before us is to be followed by a second on the same subject. In these volumes the editor aims to develop the theory of numbers in such a manner as to preserve the personal imprint of Kronecker, but to fill out the lacunæ which the lectures of this great arithmetician naturally pre-

* Under the title of "Anfangsgründe der höheren Arithmetik" Dirichlet offered a course on the theory of numbers at Berlin as early as 1833.