THE SYNTHETIC TREATMENT OF CONICS AT THE PRESENT TIME.

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IN the August-September number of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* for the year 1902 Professor Reye has printed an address, delivered on entering upon the rectorship of the university at Strassburg, on "Synthetic geometry in ancient and modern times," in which he sketches in a general manner, as the nature of his audience necessitated, the differences between the methods and points of view in vogue in geometry some two thousand years ago and at the present day. The range of the present essay is by no means so extended. It covers merely the different methods which are available at the present time for the development of the elements of synthetic geometry — that is the theory of the conic sections and its relation to poles and polars.

There is one method of treatment which was much in use, the world over, half a century ago. At the present time this method is perhaps best represented by Cremona's Projective Geometry, translated into English by Leudesdorf and published by the Oxford University Press, or by Russell's Pure Geometry from the same press. This treatment depends essentially upon the conception and numerical measure of the double or cross ratio. It starts from the knowledge the student already possesses of Euclid's Elements. This was the method of Cremona, of Steiner, and of Chasles, as set forth in their books. That was fifty years ago.

In the year 1866 Professor Reye, mixing with a bold and somewhat original hand the newest ideas of von Staudt with those of Steiner and Poncelet, older and better known, published his Geometrie der Lage which owing to its incomparably clear and pedagogic style has been the mainstay of synthetic projective geometry these many years, slowly displacing the method of Cremona, and somewhat obscuring the elegance of von Staudt's.

At present, to judge from the text-books upon the market and the announcements of courses in the universities, the Italians have not only forsaken Cremona but are persistently edging by Reye toward the long neglected von Staudt; the Germans, pulled with almost equal forces backward by Steiner and forward by von Staudt, still cling to the intermediate, Reye. In France, where no one studies that which there is not some one great genius to teach, synthetic geometry in its higher aspects has seemingly perished by the way with Chasles — for what was it to Hermite and Bertrand or is it now to Darboux and Picard? In England and America Cremona is still much followed, and when not Cremona, then Reye.

In view of these facts, no more will be said of the metric, cross ratio method of developing synthetic geometry. That method caught its fatal chill when von Staudt published his book in 1847. It has been shaking ever since. Some day it will die. Not long hence even the fondest of us must recognize it as quite a thing of the past.

The newer methods are all founded ultimately upon the work of von Staudt, the essential part of which appeared in 1847 as has been mentioned, although the addition of three supplements stretched across the following thirteen years. These methods are more powerful and beautiful because they deal more directly with the ideas necessarily involved in synthetic projective geometry. They are more natural and satisfactory from a purely geometric standpoint. They are more compact and at the same time they are more far-reaching. In point of fact they are all nothing but variations on von Staudt's original, accoiding as it is more or less fully adhered to, according as a less or a greater amount of the earlier methods are interwoven with it.

The chief variations, those which we shall discuss here, are three. We shall designate them as follows:

1° The extreme, which may be seen expounded in Reye's "Geometrie der Lage," 4th edition, volume I, Leipzig, or in Holgate's translation, New York.

2° The intermediate of which an excellent and elementary exposition is given in Böger's "Ebene Geometrie der Lage," Leipzig, or combined with the geometry of space in Sannia's "Geometria proiettiva," Naples.

3° The slight variation, or none at all, such as is presented in Enriques's most excellent "Geometria proiettiva," Bologna, or to a much less extent in a paper in the *Annals of Mathematics,* second series, volume 2, by Miss C. A. Scott.

By a glance at the books mentioned, these three methods may be seen to have in common a certain considerable number of elementary ideas and theorems; namely, the law of duality, the theorem of Desargues, the construction of harmonic elements, the theory of the projective relations between the points in two ranges or in the same range counted twice. These are explained perhaps most simply and succinctly by Böger. To them may be added, if desired, the elements of the theory of collineations and correlations between the points and lines of two planes or of the same plane counted twice.

From this point on, the three methods diverge. In the first a conic is defined as the locus of the intersection of corresponding rays in two projective, but not perspective, pencils. The usual theorems concerning inscribed and circumscribed quadrilaterals, Pascal's and Brianchon's theorems, and so forth, are then proved. The polar of a point with respect to a conic is defined by means of the well known harmonic property. The theory of poles and polars follows, completing the usual elements of conic sections from the synthetic standpoint.

The advantages of this method, the reasons why it holds its popularity so easily, are not hard to see. The processes seem direct. The construction of a conic is a pretty problem, a pretty exercise in draughting, for the student. (It is however by no means so pretty a problem to prove that a conic is independent of the two particular points upon it which may have been chosen as the vertices of the defining projective pencils.) A ruler and pencil are real, tangible things which are good for the student to use constantly. Moreover the idea of polar, introduced by means of the harmonic property, is easily grasped. The thing itself is easily constructed.

But there are disadvantages, too, of which one is that if the pole be outside the conic the construction for the polar yields only a segment of a line, only the part of the polar inside the conic. This is inconvenient. It introduces a sort of lack of symmetry which sticks out awkwardly in some proofs unless, drawing on the analogy furnished from analytic geometry, we make a bold application of the so-called principle of continuity. Again there is the awkward lack of symmetry mentioned in the parenthesis above. There are other disadvantages which will appear when the advantages of the third method are pointed out.

In the second method the conics are defined asbefore. To define the relations of pole and polar, the conception of an *involution* upon a conic is used. Points $A, B, \cdots; A', B', \cdots$ upon a conic are said to be in involution if the relation between them is projective in such a manner that if A corresponds to A', then A' corresponds mutually to A. It is shown that the lines which join corresponding points of an involution meet in a point P and that tangents drawn at corresponding points meet in a line p. The point P and line p are defined as pole and polar.

This method is not much better than the first. It is but a timorous intermediate between it and the third. It has nothing to recommend it but the drilling of the student in the ideas of projectivities and in the special case of involutions. These statements must not be construed as in any way detrimental to Böger's book, which is, on the contrary, far from tiresome and contains in its later portions a complete presentation of the newest ideas in adjoined and resultant involutions and in polar fields. In fact the standpoint, except at the commencement, is that of the third method of treatment.

The third method sets about matters in a wholly different way. Proceeding from the common ground before mentioned it lays great stress on the idea of projective relations between ranges or pencils and of collinear and correlative relations between two planes whether different or coincident. Here as in the first method the student may have plenty of use for his ruler and pencil. The subject only is different. Here he takes four points A, B, C, D and four other points A', B', C', D' and constructs corresponding points P and P' in the collineations thus defined, or he takes four lines a, b, c, d and constructs correlative points and lines P and p or inversely q and Q. These exercises are but preliminary. The important case is that of a *correlation which is involutory*, in which if P is correlative to p, then inversely p is correlative to P. Such an involutory correlation is called a polarity. The correlative points and lines are the poles and polars of the polarity.

As yet conics have not been defined. The definition is given as follows: The locus of those points in the polarity which lie upon their own polars (*if such points exist*) is a conic. Or the envelope of those lines in the polarity which pass through their own poles (if such line exist) is a conic. These two definitions lead to the same curve. The demonstrations that this curve is a conic according to the other definition of methods 1° and 2°, that the poles and polars with regard to the polarity are poles and polars with regard to the conic, and all the usual theorems concerning conics and poles and polars follow immediately with great directness and simplicity.

The disadvantage which has been attributed especially to this method is its difficulty. There certainly is difficulty in the method either for the student or for the instructor. If the latter is content to follow hard in the footsteps of von Staudt or even directly after Enriques or C. A. Scott the student will have difficulty. This method has not been so trampled smooth, so laid out with problems as the older. But if the instructor will only apply himself to obtaining graded exercises for the student, if he will be content to go slowly and carefully, he will find that in the end he will have covered as much ground as easily and comprehensibly to the student, as if he had kept to either of the other methods.

And the advantages, whether on the score of beauty and power or on that of the insight afforded to the student into the subject of projective geometry, are alike great. For it must have been observed that the fundamental difference in the three methods is the relative importance in them of the idea of the transformation—the projective transformation, whether collineation or correlation—and in particular the involutory projective transformation. In the first method only so much knowledge of transformations is needed as will enable one to associate corresponding rays in two pencils, in the second the elementary properties of involutions on a conic are required, in the third the important fundamental notions of collineation and correlation are indispensable.

For the last thirty years, or to be more precise, since the publication of Klein's Erlangen Programm in 1872, there has been no reason why geometers should not recognize distinctly — although they have often been slow to do so — that the fundamental thing, next to the axiom, is the transformation or groups of transformations. In projective plane geometry, whether the treatment be analytic or synthetic, the important thing is the group of collineations G_8 plus the correlations Γ_8. The conic, a projective concept in itself, belongs essentially to this group and may be treated most appropriately in connection with it. Von Staudt's great achievement was to see this and publish it in 1847, a quarter century before the appearance of Klein's Erlangen Programm. It is for the geometers who follow him so to arrange his method that it shall be easily accessible to students without being changed so essentially as to lose its chief vantage points.

We may note in passing that in other fields than projective geometry as much if not greater sloth in adopting the transformation theory and its benefits is exhibited. How many a weary page of Euclid's elements could be brightened by the introduction of the theory of displacements in connection with polygons and polyhedra and of the theory of inversion in connection with circles and spheres. How many a useless proposition could be replaced by one full of promise for future applications.

To return to projective geometry. One advantage of von Staudt's method has been seen to be that it places in evidence the importance of the projective group. There is another advantage which might at first seem a disadvantage. It is this. The conic has been defined subject to its existence. There is a large class of polarities which yield no conic. Analytically they would yield an imaginary ellipse of the type

$$a^2x^2 + b^2y^2 + c^2 = 0.$$

But in these polarities a great number of theorems, real in every detail, hold just as in polarities which yield a real conic. For example, the theorem that the six vertices of two triangles self-conjugate in the same polarity lie on a conic is evidently real and valid for any sort of polarity. So with other theorems.

It is evident, therefore, that the third method, which in all essentials is von Staudt's, affords a treatment of polarities and consequently of conic sections which is independent of the reality or non-reality of those conic sections, provided only that the polar field defined by them is real. This means that we have a synthetic method for treating any conic section which would be representable analytically by an equation with real coefficients. Not often is a greater generality assumed for analytic methods.

The salient advantages of von Staudt's method are then these: First, insistence on treating projective geometry from the standpoint of the projective group. Second, ability to treat in a similar manner all conics, real or imaginary, which define real polar fields. The disadvantages are a supposed difficulty and indirectness. The latter does not exist and the former would reduce almost to nothing if half the attention were expended on the best method which has been spent on the worse.

I say the best method, referring to von Staudt's. It certainly seems at present the best not only of those known, but of any which may be invented. For what more or better could be expected of any purely synthetic method than that which is afforded by the two above mentioned salient advantages of this. It seems scarcely possible that any synthetic method should treat a more general class of conics or treat them in a more concise and germane manner.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
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BROWN'S LUNAR THEORY.

An Introductory Treatise on the Lunar Theory. By ERNEST W. BROWN, M.A. Cambridge, Eng., The University Press, 1896. xvi + 292 pp.

THE lunar theory, besides containing inherent difficulties of a very serious character, involves such a mass of details that it is only with a great deal of labor that one can get a satisfactory idea of it from the original memoirs. The essential relations and differences of the various methods are obscured as they come from the hands of their authors by the differences in the choice of variables and notations. In view of the intrinsic value of the subject and of its great importance as the best test of the newtonian law, and the fact that advances in it can hardly be hoped for from one who is not familiar with what has been done in the past, the desirability of a treatise starting at the very foundations, pointing out the difficulties which are encountered and the methods which have been used to overcome them, and giving the essentials of the most important processes employed by the various investigators without including the almost endless details, can easily be appreciated. This book has evidently been written to fill this need, and it may be said at once that Professor Brown has attained a very high degree of success.

The only other book of at all the same character is the third volume of Tisserand's Mécanique céleste, which is devoted to the theory of the motion of the moon. Tisserand's ideal is somewhat different, in that he aims to give a fairly complete account of all the lunar theories of particular merit which have