I say the best method, referring to von Staudt's. It certainly seems at present the best not only of those known, but of any which may be invented. For what more or better could be expected of any purely synthetic method than that which is afforded by the two above mentioned salient advantages of this. It seems scarcely possible that any synthetic method should treat a more general class of conics or treat them in a more concise and germane manner.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
November, 1902.

BROWN'S LUNAR THEORY.

An Introductory Treatise on the Lunar Theory. By ERNEST W. BROWN, M.A. Cambridge, Eng., The University Press, 1896. xvi + 292 pp.

THE lunar theory, besides containing inherent difficulties of a very serious character, involves such a mass of details that it is only with a great deal of labor that one can get a satisfactory idea of it from the original memoirs. The essential relations and differences of the various methods are obscured as they come from the hands of their authors by the differences in the choice of variables and notations. In view of the intrinsic value of the subject and of its great importance as the best test of the newtonian law, and the fact that advances in it can hardly be hoped for from one who is not familiar with what has been done in the past, the desirability of a treatise starting at the very foundations, pointing out the difficulties which are encountered and the methods which have been used to overcome them, and giving the essentials of the most important processes employed by the various investigators without including the almost endless details, can easily be appreciated. This book has evidently been written to fill this need, and it may be said at once that Professor Brown has attained a very high degree of success.

The only other book of at all the same character is the third volume of Tisserand's Mécanique céleste, which is devoted to the theory of the motion of the moon. Tisserand's ideal is somewhat different, in that he aims to give a fairly complete account of all the lunar theories of particular merit which have been developed. Moreover, his third volume is not complete in itself, as the results of the first two volumes are used so far as they are needed.

An idea of the contents and arrangement of Professor Brown's treatise can be obtained from the titles of the chapters, which are in their order: Force functions, The equations of motion, Undisturbed elliptic motion, Form of solution, The first approximation, Variation of arbitrary constants, The disturbing function, De Pontécoulant's method, The constants and their interpretation, The theory of Delaunay, The method of Hansen, Method with rectangular coördinates, The principal methods, Planetary and other disturbing influences. In the table of contents a complete enumeration of the topics treated is given, and this is supplemented by a general index, an index of authors, a reference table of notation, a general scheme of notation, and a comparative table of notation. From this it is seen that the work has been arranged so as to make its reading as simple as possible for the beginner, and to make it a valuable reference work for those who may be familiar with the subject.

The care with which the preliminary ground has been gone over has put the actual introduction of the perturbations due to the sun far along in the book. Thus, the reader finds himself compelled to read six chapters before he gets to that which he is continually expecting. It might, perhaps, have been better to have put the long chapter on the variation of constants after the chapters on de Pontécoulant's method and the constants and their interpretation. This would have remedied to a considerable extent that which in any case is apt to strain the patience of the reader. Aside from the possible modification just mentioned, the general plan of the book could scarcely be improved.

There are three essentially different types of lunar theory — that of de Pontécoulant, that of Delaunay, and that first developed by Hill, to which may perhaps be added that of Hansen as containing many features of more or less importance different from the others. That of de Pontécoulant and most of his predecessors consists in developing certain coördinates in periodic series of assumed form with the time or true anomaly as argument and determining the coefficients step by step as powers of the small parameters involved; that of Delaunay consists in applying the method of the variation of parameters in the canonical form over and over in such a way as to remove the most important parts of the perturbative function; that of Hill consists in finding very accurate particular solutions of the differential equations after the parts depending on the parallax of the sun, the eccentricity of the earth's orbit, and the latitude of the moon have been neglected, and then finding the deviations from this orbit due to general initial conditions and the neglected part of the perturbative function. The author has wisely treated these three types and the method of Hansen in detail, and explained the others briefly by their relations to them.

A number of typographical errors have crept in, but this could scarcely be avoided in a book involving so many complicated formulas. Those which have been noticed will cause the reader no trouble. There are some other points to be mentioned, having their origin for the most part in the imperfect state of the lunar theory from a mathematical standpoint rather than in faults of presentation of well-established methods. Thus, in the exposition of de Pontécoulant's method, which is somewhat different from the original yet fairly representative of methods of this type, the treatment of the constants of integration leaves something to be desired. The differential equations for motion in the plane of the ecliptic are, if $r = 1/u$ is the radius of the moon's orbit and v her longitude,

$$\frac{\frac{1}{2}d^2\left(\frac{1}{u^2}\right)}{dt^2} - u + \frac{1}{a} = m^2 P_1(u, v, e', t),$$

$$\frac{dv}{dt} - hu^2 = m^2 P_2(u, v, e', t).$$

a and h are constants depending on the initial conditions, m is the ratio of the mean motion of the sun to that of the moon, and e' is the eccentricity of the earth's orbit. If the right members are neglected, ordinary elliptic motion is obtained. In this, u is a periodic series whose coefficients are polynomials in the eccentricity of the moon's orbit, their structure depending uniquely on the form of the differential equations.

If the coördinates have the form $u = u_0 + \delta u$, $v = v_0 + \delta v$, where δu and δv are the perturbations, it is found after substituting in the differential equations that the main part of δu is defined by the equation

$$\frac{1}{n^2}\frac{d^2\delta u}{dt^2} + \delta u = m^2 Q(t).$$

The general solution of this linear equation consists of two parts, the particular integral and the complementary function. The period of the complementary function is the same as that of the first periodic term in the elliptic motion. For this reason (see pages 50 and 117) the complementary function is omitted, or rather is supposed to be included in the elliptic term. This is of course allowable since the coefficient of the first elliptic term is arbitrary, but it is not evident that the coefficients of the higher multiples of the elliptic argument are related to this *modified* coefficient of the first term precisely as they are in undisturbed motion. This seems to be assumed tacitly, and it is to be regretted that the reason for it is not given if it is true. The fact is that there are but six arbitrary constants in the solutions of a system of differential equations of the sixth order, and it should be clearly shown how all of the constants can be expressed in terms of the six which are taken as the arbitraries. Some of the statements in Chapter VII might lead a careless reader to infer that many constants may be taken arbitrarily. It would have been clearer if the statement had been made that the constants $E + B + \cdots$ and similar combinations should in the final results be set equal to a constant e_1 rather than that any of them should arbitrarily be set equal to zero. Such a method would express all of the constants in terms of six arbitraries; at the end transformations would be made so as to give six coefficients any desired form. This is the method clearly carried out by Delaunay.

Chapter VIII, which is devoted to a discussion of the meaning of the constants (that is, the coefficients of the principal terms) and the manner in which they are determined, is much to be commended. The treatment of the methods of Delaunay and Hansen is also quite satisfactory.

Chapter XI treats of the methods inaugurated by Hill in his celebrated Researches, and of the extensions to the terms of higher order which the author has himself carried out in numerous important memoirs. It is unfortunately called the "Method with rectangular coördinates." In the first place all of the earlier methods are named after their respective authors and there is no apparent reason why it should not be done here, especially in view of the fact that its whole spirit is radically different from that of any previous method. In the second place the title adopted in no fundamental way characterizes the method. It is true that the differential equations are at first written in rectangular coördinates where the axes rotate uniformly ; but before the integration is begun a transformation is made to complex variables. It is clear that the rectangular coördinates are only incidental and that the final equations in the complex variables might be derived without using them at all. Hill's first problem was to find very exact particular solutions of the chief parts of the differential equations, and in doing this he used two important artifices. The first was to remove the explicit presence of the time in the differential equations by rotating the axes ; the second, to reduce the differential equations to the bilinear form by the use of complex variables in order that the task of finding the coefficients of the solutions might not be hopelessly involved. There is no trouble in finding the coefficients when the solutions of differential equations are developed as power series in the independent variable, or in parameters, but the general problem of finding the coefficients when the solutions are Fourier series presents great difficulties. Hill overcame them with rare ingenuity in this problem. If it is desired to give the method a title which characterizes it the method of the variational orbit might be used, but the reviewer suggests as being more appropriate that it be called the method of Hill, or the Hill-Brown method.

In developing the particular solutions of the restricted differential equations Brown, and Tisserand also, infer their form from the results of the earlier methods in the lunar theory. These methods, besides not having been proved to give convergent series, lead to expressions of the same form for general initial conditions. Now the particular solutions found depend upon particular initial conditions for their existence. Hill correctly inferred their character, if they exist, from the form of the differential equations; and he proved their existence by the actual construction of certain nearly periodic orbits by mechanical quadratures and an appeal to the analytic continuity of curves when considered as functions of the initial values of the variables. It seems clear that Brown and Tisserand have taken a backward step in this matter, for it is evident that assumptions should be introduced as sparingly as possible; and that an ideal method is complete in itself.

The discussion of the terms which depend upon the ratio of the periods of the moon and sun and the first power of the eccentricity of the moon's orbit is in excellent shape for one to get from it a practical command of the ideas which Hill developed in his memoir in the *Acta Mathematica.* Yet from a theoretical point of view it would have been better to have proved the existence of the periodic solutions and their form, or at least to have referred to places where such proofs are given, rather than to have inferred their existence and character from earlier theories. The equations which give these terms are linear and non-homogeneous with periodic coefficients and the properties of their solutions are now well known. It might also have been remarked that there are other methods of finding the coefficients and the constant which determines the part of the motion of the moon's perigee which is of the first order in the eccentricity.

A very interesting phenomenon occurs in the literal expansions of the coefficients of the variational terms. If the expansions are made in powers of m (m equals about $\frac{1}{13}$), where m is the ratio of the mean motion of the sun to that of the moon, the series converge very slowly. But if the expansions are made in powers of m' where $m' = m/(1 - m/3)$ (therefore $m' = \frac{3}{38}$ nearly) the series converge with great rapidity. It is at first astonishing that so slight a change in the parameter should have such important results.

The most important parts of these coefficients come from the expansions of $(6 - 4m + m^2)^{-1}$. Hill inquires what value of α must be substituted in the equation $m' = m/(1 + \alpha m)$ so that the series in m' shall converge most rapidly. Hill says, and Brown repeats the statement, that it is easily found that α must be given the value $-\frac{1}{3}$. It is not stated just what is meant by rapidity of convergence, but if the remainder after a given number of terms is to be a minimum the statement is not perfectly correct, although $-\frac{1}{3}$ is undoubtedly the most convenient value to give α in the case in hand. If m had a value large enough the series would actually converge more slowly after the transformation.

The radius of convergence of the series in m is $\sqrt{6}$, and in m', when $\alpha = -\frac{1}{3}$, $\sqrt{18}$. If m has such a value that $m' > \sqrt{3}\, m$ the new series will converge less rapidly than the old. The value of α which must be used to secure a minimum for the remainder depends upon both the numerical value of m and the number of terms taken. Its value can be determined nearly from the formulas for the remainder limited by inequalities.

Let $f(m) = (6 - 4m + m^2)^{-1}$ and $\phi(m')$ be the same function after the substitution $m' = m/(1 + \alpha m)$. Let ρ represent the true radius of convergence of $f(m)$ when expanded as a power series in m, and ρ' the corresponding radius when $\phi(m')$ is expanded as a power series in m'. Let ρ_1 be a real positive number such that $\rho_1 < \rho$. Let M be the maximum value of the function $f(m)$ on the circumference of the circle whose radius is ρ_1. As m takes all the values on the circumference of this circle m' will take all the values on the circumference of a circle whose radius may be represented by ρ_1'. The maximum value of $\phi(m')$ on this circumference is also M. Let $R_n(m')$ be the remainder of the series in m' after the first n terms have been taken. Then $R_n(m')$ is limited by the inequality

$$R_n(m') < \frac{M\left(\frac{m'}{\rho'}\right)^n}{1 - \frac{m'}{\rho'}} \text{ for all } |m'| < \rho_1'.$$

Consider now the problem of making $R_n(m')$ a minimum. Let ρ_1 be taken as the numerical value of m; hence M is defined by the given value of m and the form of the function $f(m)$, and is a constant with respect to the proposed transformation. Without an absolute minimum being insured because of the presence of the inequality, $R_n(m')$ will generally be smaller the smaller the fraction $(m'/\rho')^n/(1 - m'/\rho')$ is made. This fraction depends upon α as the arbitrary. Let it be represented by $F(\alpha)$. The necessary condition for a minimum is $\partial F(\alpha)/\partial\alpha = 0$. In order to compute this derivative, m' and ρ' must be expressed in terms of α. From the equation of transformation $m' = m/(1 + \alpha m)$ and therefore $m = m'/(1 - \alpha m')$ and

$$(6 - 4m + m^2)^{-1}$$
$$= (1 - \alpha m')^2[6 - 4(1 + 3\alpha)m' + (1 + 4\alpha + 6\,\alpha^2)m'^2]^{-1}.$$

The only singularities are poles and it follows at once that

$$\rho' = \frac{\sqrt{6}}{\sqrt{1 + 4\alpha + 6\alpha^2}}.$$

Now forming the partial derivative, casting out the common factors, and clearing of fractions it is found that

$$12n(1+3\alpha)(1+\alpha m)^2-6nm(1+\alpha m)(1+4\alpha+6\alpha^2)$$
$$-2\sqrt{6}m(n-1)(1+3\alpha)(1+\alpha m)\sqrt{1+4\alpha+6\alpha^2}$$
$$+\sqrt{6}(n-1)m(1+4\alpha+6\alpha^2)^{\frac{3}{2}}=0.$$

When $m=0$ then $\alpha+\frac{1}{3}=0$ is a simple solution of this equation. Therefore α may be represented as a power series of the form

$$\alpha=-\tfrac{1}{3}+a_1m+a_2m^2+\cdots$$

which converges for sufficiently small values of m. Substituting this expression in the preceding equation and equating to zero the coefficients of the various powers of m it is found that

$$a_1=\tfrac{1}{18}-\frac{\sqrt{2}(n-1)}{108},$$

....

whence

$$\alpha=-\tfrac{1}{3}+\frac{m}{18}-\frac{\sqrt{2}(n-1)\,m}{108}+\cdots.$$

Consequently $-\frac{1}{3}$ is the limit of the value of α as m approaches zero. As m is very small in the lunar theory this value of α is very satisfactory, and particularly so as it changes the trinomial into a binomial.

To illustrate the matter consider a numerical example with a large value of m, say $m=\frac{1}{2}$. Then let the remainder be considered after three terms are taken, or $n-1=3$. It follows that $\alpha=-.318650$, $m'=.594760$, and $(6-4m+m^2)=.235294$. Now it is easily found that

$$[6-4m+m^2]^{-1}=\tfrac{1}{6}(1+\tfrac{2}{3}m+\tfrac{5}{18}m^2)+R_3(m),$$

and when $\alpha=-\frac{1}{3}$

$$[6-4m+m^2]^{-1}=\tfrac{1}{6}(1+\tfrac{2}{3}m'+\tfrac{1}{18}m'^2)+R_3'(m'),$$

and when $\alpha=-.318650$

$$[6-4m+m^2]^{-1}=\tfrac{1}{6}(1+\tfrac{2}{3}m'+.065345m'^2)+R_3(m').$$

It is found by substituting the numerical values that

$$R_3(m) = .001498,$$
$$R'_3(m') = -.001373,$$
$$R_3(m') = -.001310.$$

These few terms are only enough to illustrate the matter.

The object of a literal expansion of the coefficients is doubtless to enable one to compute them most conveniently for various values of the parameter which may arise in practice. In the lunar theory, where alone there is any demand for extended computations, the parameter m is known with a high degree of accuracy. If it were exactly known there would be no occasion for these expansions for use in the lunar theory, since the power series possess no theoretical advantages over the series of fractions either when retained literally or when reduced to numbers.

If the numerical value is given to m on the start the labor of computing the coefficients is enormously reduced, but they admit of no direct corrections for improved values of the parameter. But if m_0 represents the approximate value of m which is at present known, and if the substitution $m = m_0 + m'$ be made, a series will be obtained in m' which will converge extremely rapidly because of the smallness of this parameter. A very few terms will be sufficient to give the results correctly to many decimals for any value of m' which is apt to appear necessary to use. For practical application in the lunar theory this preserves all the generality attained in using m and secures most of the convenience of using numbers from the start. The unfavorable features are that the numerical coefficients cannot be expressed as ordinary fractions and the trinomial $(6 - 4m + m^2)$ remains a trinomial after the transformation. However, if the substitution

$$m = \frac{m_0 + m'}{1 + am'}$$

is used, the α may be determined so that the trinomial shall reduce to a binomial after which the expansion may be made with great ease. For values of m far different from m_0 the series in m' converge about as rapidly as those in m.

Chapters XII and XIII, which treat briefly of the principal theories other than those considered earlier in the text and of the effects of planetary and other disturbing influences, are much to be commended except the paragraph in the center of the last page. In this the astonishing statement is made that, because the moon's orbit shrinks as the eccentricity of the earth's orbit decreases, the moon would fall on the earth if the earth's orbit became circular. Obviously a variable may always decrease without having zero as its limit, and in this case the incorrectness of the conclusion is quite evident.

In recapitulation it may be said that Professor Brown has written a very satisfactory treatise on the lunar theory, the faults being for the most part those which have their origin in the inherent, and up to the present unsolved, difficulties of the subject. There is no better source of information for one who wishes to acquaint himself with this celebrated field of investigation.

F. R. MOULTON.

THE UNIVERSITY OF CHICAGO,
November, 1902.

THE DOCTRINE OF INFINITY.

Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. Von Dr. K. GEISSLER. Leipzig, Teubner, 1902. 4to, 417 pp.

THE recent tendency toward critical investigation of the axioms underlying mathematical sciences has led to the necessity of discussing the philosophical basis of the whole mathematical structure, and it is to be hoped that more attention will be given the subject from a purely philosophical standpoint.

The announcement of a book treating the old stumbling block of the infinite from both the mathematical and the philosophical points of view was therefore of the greatest interest, and the character of the book was apparently guaranteed by the excellent reputation of the publishers. But the book — though not lacking in a certain kind of interest — is in several respects most surprising, and, as we shall see, somewhat disappointing.

The first 296 pages are devoted to mathematical investigations, and we shall consider these alone. A great variety of topics are treated, such as the parallel axiom, indeterminate forms, tangents, limits, derivates, velocity, curvature and many