THE NINTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Ninth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Monday and Tuesday, December 29–30, 1902. The following sixty-two members were in attendance at the four sessions:

Dr. Grace Andrews, Mr. C. H. Ashton, Professor Joseph Bowden, Professor E. W. Brown, Dr. A. B. Chace, Professor A. S. Chessin, Dr. J. E. Clarke, Dr. A. B. Coble, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Mr. D. R. Curtiss, Dr. W. S. Dennett, Dr. Otto Dunkel, Professor T. C. Esty, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Dr. G. B. Germann, Dr. G. H. Hallett, Miss Carrie Hammerslough, Professor James Harkness, Dr. H. E. Hawkes, Dr. E. R. Hedrick, Dr. A. A. Himowich, Dr. E. V. Huntington, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Professor Pomeroy Ladue, Professor P. A. Lambert, Professor J. L. Love, Dr. Emory McClintock, Professor H. P. Manning, Professor W. H. Metzler, Mr. H. B. Mitchell, Professor E. H. Moore, Professor F. Morley, Professor G. D. Olds, Professor W. F. Osgood, Professor Anna H. Palmié, Professor James Pierpont, Professor R. W. Prentiss, Professor M. I. Pupin, Dr. I. E. Rabinovitch, Professor E. D. Roe, Jr., Professor Charlotte A. Scott, Mr. Burke Smith, Professor D. E. Smith, Professor P. F. Smith, Dr. H. F. Stecker, Professor Irving Stringham, Dr. W. M. Strong, Professor H. W. Tyler, Professor E. B. Van Vleck, Dr. Roxana H. Vivian, Professor L. A. Wait, Mr. A. C. Washburne, Professor F. S. Woods, Professor R. S. Woodward, Professor J. W. A. Young, Mr. J. W. Young.

The retiring President of the Society, Professor Eliakim Hastings Moore, presided at the earlier sessions, being relieved by Vice-Presidents Professor F. Morley and W. F. Osgood, and giving way on Tuesday to the President-Elect, Professor Thomas Scott Fiske. The Council announced the election of the following persons to membership in the Society: Dr. A. B. Coble, University of Missouri; Mr. W. A. Cornish, State Normal School, Cortland, N. Y.; Dr. A. G. Hall, University of Michigan; Mr. E. L. Hancock, Purdue University; Professor L. M. Hoskins, Stanford University; Mr. W. D. A. Westfall, Yale University; Mr. W. F. White, State Normal School, New Paltz, N. Y. Sixteen applications for admission to the Society were received.

Reports were presented by the Librarian, the Treasurer, and the Auditing Committee. These reports will appear in the Annual Register, now in press. During the past year the membership of the Society has increased from 375 to 399; the number of papers presented at the meetings from 136 to 156. The total attendance at all meetings was 266; 144 members attended at least one meeting. The library now contains about 1000 bound volumes, and nearly 120 mathematical journals are regularly received in exchange for the *Bulletin* and the *Transactions*. A catalogue of the library will be published in the Annual Register.

Coming at the expiration of the presidential term of office, the meeting was especially marked as the occasion of a presidential address. Under the title: "On the foundations of mathematics," the retiring President, Professor E. H. Moore, urged the desirability of the Society undertaking to exert an effective influence on the teaching of elementary mathematics. A committee of the Council was appointed to consider this question.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen:

President,	Professor THOMAS S. FISKE.
Vice-Presidents,	Professor W. F. OSGOOD.
	Professor ALEXANDER ZIWET.
Secretary,	Professor F. N. COLE.
Treasurer,	DR. W. S. DENNETT.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1905,

Professor JAMES HARKNESS, Professor IRVING STRINGHAM,
Professor HEINRICH MASCHKE, Professor H. W. TYLER.

The following papers were read at the annual meeting:

(1) Dr. E. V. HUNTINGTON: "A complete set of postulates for the theory of real numbers (second paper)."

(2) Dr. E. V. HUNTINGTON: "On the definitions of the elementary functions by means of definite integrals."

(3) Dr. C. J. KEYSER: "On the axiom of infinity."

(4) Professor G. H. DARWIN: "The approximate determination of the form of Maclaurin's spheroid."

(5) Professor HARRIS HANCOCK: "Remarks on the sufficient conditions in the calculus of variations."

(6) Professor L. E. DICKSON: "The abstract group G simply isomorphic with the alternating group on six letters."

(7) Professor E. H. MOORE: Presidential address: "On the foundations of mathematics."

(8) Professor W. E. TAYLOR: "On the product of an alternant and a symmetric function."

(9) Professor E. D. ROE, JR.: "On the coefficients in the product of an alternant and a symmetric function."

(10) Professor E. D. ROE, JR.: "On the coefficients in the quotient of two alternants" (preliminary communication).

(11) Professor E. O. LOVETT: "A transformation group of $(2n - 1)(n - 1)$ parameters, and its rôle in the problem of n bodies."

(12) Dr. I. E. RABINOVITCH: "On the solid lunes of conicoids, analogous to the circular lunes of Hippocrates of Chios."

(13) Dr. E. B. WILSON: "The synthetic treatment of conics at the present time."

(14) Dr. A. B. COBLE: "On the invariant theory of the connex (2, 2) of the ternary domain viewed as a connex (1, 1) in a five dimensional space."

(15) Dr. EDWARD KASNER: "The general quadratic systems of conics and quadrics."

(16) Professor W. F. OSGOOD: On the transformation of the boundary in the case of conformal mapping."

(17) Professor W. F. OSGOOD: "A Jordan curve of positive area."

(18) Mr. J. W. YOUNG: "On the automorphic functions associated with the group of character [0, 3; 2, 4, ∞]" (preliminary report).

(19) Mr. R. W. H. T. HUDSON: "The analytic theory of displacements."

(20) Dr. H. E. HAWKES: "Enumeration of the non-quaternion number systems."

(21) Dr. H. F. STECKER: "On the parameters in certain systems of geodesic lines."

(22) Dr. EDWARD KASNER: "The generalized Beltrami problem concerning geodesic representation."

(23) Professor MAXIME BÔCHER: "Singular points of functions which satisfy partial differential equations of the elliptic type."

(24) Mr. G. D. BIRKHOFF and Mr. H. S. VANDIVER: "General theory of the integral divisors of $a^n - b^n$, and allied cyclotomic forms."

(25) Professor G. A. MILLER: "A new proof of the generalized Wilson theorem."

(26) Professor F. MORLEY: "On the determinant $|(x_i - a_j)^{-2}|$."

Professor Darwin's paper was communicated to the Society through Professor E. W. Brown, Professor Taylor's through Professor Metzler, and Mr. Hudson's through Professor Morley. Mr. Vandiver was introduced by Professor Crawley. In the absence of the authors, Professor Darwin's paper was read by Professor Brown, Professor Taylor's by Professor Roe, Professor Bôcher's by Professor Osgood, and the papers of Professor Hancock, Professor Dickson, Professor Lovett, Dr. Wilson, Mr. Hudson and Professor Miller were read by title.

Dr. Wilson's paper and Professor Osgood's first paper appeared in the February BULLETIN. The papers of Professor Dickson and Mr. Hudson are contained in the present number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. The two sets of postulates for real number * contained in Dr. Huntington's present paper are more convenient and natural than the set proposed by him at the April meeting (see abstract in BULLETIN, volume 8, page 371). The fundamental concepts involved in the present paper are 1° a *relation*, $\otimes$, in which one element may stand to another — $a \otimes b$ being read: a "less than" b; 2° a *rule of combination*, $\oplus$, according to which every two elements a and b of a given assemblage de-

*Cf. D. Hilbert, *Jahresber. d. Deutschen Math.-Vereinigung*, volume 8 (1900), page 180.

termine uniquely an object $a \oplus b$ called their "sum"; and 3° another *rule of combination*, $\odot$, according to which a and b determine uniquely an object $a \odot b$ called their "product." Any assemblage in which $\otimes$ and $\oplus$ are so defined as to satisfy the nine postulates of the first set below, or any assemblage in which $\otimes$, $\oplus$ and $\odot$ are so defined as to satisfy the thirteen postulates of the second set, will be equivalent to the system of all real numbers (positive, negative and zero). Each set is "complete"; that is, the postulates of each set are (a) consistent, (b) independent, and (c) sufficient to define essentially a single assemblage.

The first set contains the following postulates:

1. If $a \neq b$ then either $a \otimes b$ or $b \otimes a$.
2. If $a \otimes b$ and $b \otimes c$ then $a \otimes c$.
3. The relation $c \otimes c$ is false for at least one element c.
4. If $a \otimes b$ then there is an element x such that $a \otimes x$ and $x \otimes b$.
5. If S is an infinite sequence of elements a_k such that

$$a_k \otimes a_{k+1}, \quad a_k \otimes c \qquad (k = 1, 2, 3, \cdots)$$

(where c is some fixed element), then there is an element A having the following two properties: $a_k \otimes A$ whenever a_k belongs to S; and if $A' \otimes A$ then there is an element a_r of S such that $A' \otimes a_r$.

6. If a, b and $b \oplus a$ belong to the assemblage, then $a \oplus b = b \oplus a$.
7. If a, b, c, $a \oplus b$, $b \oplus c$ and $a \oplus (b \oplus c)$ belong to the assemblage, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
8. For every two elements a and b there is an element x such that $a \oplus x = b$.
9. If $x \otimes y$ then $a \oplus x \otimes a \oplus y$.

The second set contains these nine postulates and also the following four:

10. The product $a \odot b$ always belongs to the assemblage.
11. First distributive law: $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.
12. Second distributive law: $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.
13. If 0 is an element such that $c \oplus 0 = c$ for every element c, and if $0 \otimes a$ and $0 \otimes b$, than $0 \otimes a \odot b$.

2. The usual plan of defining the elementary functions first for real variables and afterwards, by a process of generalization, for complex variables, involves much unnecessary repetition in

the proofs of many of their properties. In Dr. Huntington's second paper the attempt is made to avoid this repetition by defining the functions once for all for the most general case. In the first place, the fundamental theorems concerning definite integrals along a path in the complex plane (especially Goursat's theorem that the integral of a differentiable function around a closed path is zero) are rigorously developed without presupposing any knowledge of functions of a real variable, except addition, subtraction, multiplication and division. Next, $\log z$ is defined as the integral of dz/z from 1 to z, and the fundamental relation $\log a + \log b = \log (ab)$ obtained by a familiar device.* Then e^z is defined as the inverse of $\log z$, and x^y as $e^{y \log x}$. The existence and properties of these functions follow immediately for the most general case, without separate consideration of irrational and imaginary values. Finally, the integration of such functions as $1/(1 + z^2)$ and $1/\sqrt{1 - z^2}$ leads very naturally to the definitions of $\tan^{-1} z$ and $\sin^{-1} z$ as abbreviations for certain logarithmic expressions, while $\tan z$ and $\sin z$ appear as the inverses of $\tan^{-1} z$ and $\sin^{-1} z$ respectively.

3. Dr. Keyser's paper undertakes a critical examination (1) of Dedekind's "theorem of complete induction," and (2) of the proofs offered by Dedekind and Bolzano of the so-called existence theorem for infinite assemblages. In respect to (1), it is shown that that theorem, so far from being "*the* scientific basis of the method of argument called complete induction," can itself be proved by means of mathematical induction provided one recognizes that our faith in the validity of, in the apodictic certainty yielded by, this argument form finds adequate justification in a certain a priori judgment (apparently first formulated as such by Poincaré), here called the axiom of infinity and shown to be a presupposition of all logical discourse. It follows that against all such "proofs" as are referred to under (2) there must lie the charge of circularity. In course of the article several theorems in the doctrine of Ketten are established.

4. The following extract indicates the nature of Professor Darwin's paper, which will appear in the *Transactions:*

* Cf. H. Burkhardt, Analytische Functionen, p. 162, and an article by J. W. Bradshaw about to appear in the *Annals of Mathematics*.

"Spherical harmonics render the approximate determination of the figure of a rotating mass of liquid a very simple problem. If ρ be the density, e the ellipticity, and ω the angular velocity of the spheroid, the solution is

$$\frac{\omega^2}{2\pi\rho} = \frac{8}{15}e.$$

This result is only correct as far as the first power of the ellipticity, but M. Poincaré has recently shown (*Philosophical Transactions*, volume 198 *A*, pages 333–373) how harmonic analysis may be used so as to give results as far as the squares of small quantities; and I have myself used his suggestion for the determination of the stability of the pear-shaped figure of equilibrium (*Philosophical Transactions*, volume in the press).

Both these papers involved the use of ellipsoidal harmonic analysis and it would be rather tiresome to extract the method from the complex analysis in which it is embedded. It therefore seems worth while to treat the well-worn subject of Maclaurin's spheroid as an example of the method in question."

5. Professor Hancock's paper is in summary as follows. In order that the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt,$$

where F is a regular one-valued function of its arguments, may have a maximum or minimum value, it is necessary that the function F have the two properties: 1° It must be homogeneous of the first degree in x' and y'; 2° The partial derivatives $\partial F/\partial x'$ and $\partial F/\partial y'$ must be continuous, even if there are sudden changes in x' and y'. We have further the four conditions: 1° The differential equation $G = 0$ must be satisfied for every point of the curve within the interval $t_0 \cdots t_1$; 2° The function F_1 must retain the same sign within the interval, and cannot become zero or infinite within this interval; 3° There cannot be conjugate points within this interval; 4° The function $E(x, y, p, q, \bar{p}, \bar{q})$ must always have the same sign within the interval $t_0 \cdots t_1$. The conditions 2° and 3° are necessary to establish the condition 4°.

8. The object of Professor Taylor's paper is, in extension of a problem started by Muir, to determine independently the co-

efficient of any alternant occurring in the product of a symmetric function of weight w and the alternant $|0123 \cdots n-1|$. Proceeding from the standpoint of Muir, the symmetric function is regarded as being first expressed in the usual manner as a sum of products of elementary symmetric functions of weight w with certain coefficients. The problem then becomes, to find the coefficient of any alternant in the product of the single alternant and a product of elementary symmetric functions of weight w. A general reduction formula for such a coefficient is found. In particular the coefficient of the alternants of the type $|0123 \cdots r_1 s_1 \cdots r_2 s_2 \cdots r_3 s_3|$, where $s_1 - r_1 = 2$, $s_2 - r_2 = 2$, $s_3 - r_3 = q$, in the product $| 012 \cdots n-1 | \sum a_1 a_2 \cdots a_{n-i} (\sum a_1)^i$ is found to be

$$\frac{i(i-1)\cdots(i-s_1+2)}{\underline{|s_1-1}} \cdot \frac{(s_1-2)(s_1-3)\cdots(s_1+r_3-s_3-2)}{\underline{|s_3-r_3-1}}.$$

If tables of coefficients for weight w be made, others of the same type for weight $w+j_h$ can be constructed by means of the preceding reduction formula. Tables of this kind from weight one to weight seven were given. The coefficient of any alternant in the product of the original symmetric function and $| 0123 \cdots (n-1) |$ can then be obtained by multiplying the coefficient of the table of weight w by the coefficient of the elementary symmetric function product in the symmetric function table, and by taking the sum of all such products.

9. In Professor Roe's paper, which is an extension of a previous one by the author, the direct product of the symmetric function $\sum_1' \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}$ and the alternant $| \lambda_1 \lambda_2 \cdots \lambda_m |$ is considered. The coefficient of any alternant $| \kappa_1 \kappa_2 \cdots \kappa_m |$ in the product is first shown to be equal to the coefficient of $\alpha_{p_1} \alpha_{p_2} \cdots \alpha_{p_m}$ in the determinant

$$\begin{vmatrix} a_{\kappa_1-\lambda_1} & a_{\kappa_2-\lambda_1} & \cdots & a_{\kappa_m-\lambda_1} \\ a_{\kappa_1-\lambda_2} & a_{\kappa_2-\lambda_2} & \cdots & a_{\kappa_m-\lambda_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\kappa_1-\lambda_m} & a_{\kappa_2-\lambda_m} & \cdots & a_{\kappa_m-\lambda_m} \end{vmatrix}$$

and then a general recurrence formula for the coefficient is established. A method of calculating it by means of bringing about the identity of two complexes is also explained. It is next shown how a product table for all symmetric functions of the preceding type of weight w and the alternant $|012\cdots m-1|$ can be so constructed by placing conjugate partitions in a vertical, and conjugate alternants in a horizontal line, equidistant from each end, that all the coefficients lie in a triangle whose hypotenuse consists of ones, and the sum of whose columns is always zero, excepting the last one which is unity. For this table general formulas are given for the coefficients. Moreover it is shown that the whole table is absolutely invariant and still gives the complete products if $|012\cdots m+r-1|$ is used as multiplier. It is shown how the coefficients of a quotient table are certain determinants taken from the product table, and that Dr. Taylor's *product* table is the author's *quotient* table.

As it was the author's original object to obtain a method of computing symmetric functions, it is finally shown that the table is a table for all the symmetric functions of weight w written in the vertical line.

10. In his second paper Professor Roe stated for the quotient table of alternants mentioned in his first paper, eight properties which he has already observed.

11. In Professor Lovett's paper attention is called to a group of $(2n-1)(n-1)$ transformations, in as many variables, which the author encountered in a study of the problem of n bodies, the law of force being an arbitrary function of the distance. The invariants of this group are functions of the integrals of the problem of n bodies which are independent of the law of force. This interpretation appears by choosing as the variables certain quadratic functions of the coördinates and velocities of the bodies of the system.

The paper itself is concerned with the case where n is arbitrary, but the group for the particular case $n=4$ may be presented in this brief abstract as a typical one; its twenty-one infinitesimal transformations may be written as follows:

$$X_{1,123}f \equiv 2s_{1,q_1} + r_{1,s_1} + s_{21,q_{12}} + s_{31,q_{13}} + r_{21,s_{12}} + r_{31,s_{13}},$$

$$X_{2,123}f \equiv 2s_{1,r_1} + q_{1,s_1} + s_{12,r_{12}} + s_{13,r_{13}} + q_{21,s_{21}} + q_{31,s_{31}},$$

$$X_{3,123}f \equiv 2s_{21,r_1} + 2s_{12,r_2} + q_{12,q_1} + q_{12,q_2} + q_{1,r_{12}} + q_{2,r_{12}} + s_{13,r_{23}}$$
$$+ s_{23,r_{31}} + q_{1,s_{12}} + q_{2,s_{21}} + q_{32,s_{31}} + q_{31,s_{32}},$$

$$X_{4,123}f \equiv 2q_{1,q_1} - 2r_{1,r_1} + q_{12,q_{12}} + q_{13,q_{13}} - r_{12,r_{12}} - r_{13,r_{13}} + s_{12,s_{12}} + s_{13,s_{13}} - s_{21,s_{21}} - s_{31,s_{31}},$$

$$X_{5,123}f \equiv 2q_{12,q_1} - 2r_{12,r_2} + s_{21,s_1} - s_{21,s_2} + q_{2,q_{12}} + q_{32,q_{13}} - r_{1,r_{12}} - r_{31,r_{23}} + s_{2,s_{12}} - s_{1,s_{12}} + s_{23,s_{13}} - s_{31,s_{32}},$$

$$X_{6,123}f \equiv 2q_{12,q_2} - 2r_{12,r_1} + s_{12,s_2} - s_{12,s_1} + q_{1,q_{12}} + q_{31,q_{23}} - r_{2,r_{12}} - r_{32,r_{31}} + s_{1,s_{21}} - s_{2,s_{21}} + s_{13,s_{23}} - s_{32,s_{31}},$$

$$X_{7,123}f \equiv 2s_{12,q_1} + 2s_{21,q_2} + r_{12,s_1} + r_{12,s_2} + s_{1,q_{12}} + s_{2,q_{12}} + s_{31,q_{32}} + s_{32,q_{31}} + r_{2,s_{12}} + r_{1,s_{21}} + r_{23,s_{13}} + r_{31,s_{23}},$$

$$X_{i,231}f, \quad X_{i,312}f, \quad (i = 1, 2, 3, \cdots, 7),$$

where $a_{i,b_j} \equiv a_i \partial f/\partial b_j$; $q_{ij} = q_{ji}$; $r_{ij} = r_{ji}$; $s_{ij} \neq s_{ji}$.

12. Dr. Rabinovitch considered the surface of revolution $f(\sqrt{x^2 + y^2}, z) = 0$, referred to rectangular coördinates, the axis of z being the axis of revolution. The meridian planes (1) $y = 0$ and (2) $y = mx$, will cut each parallel circle $z = k$, $x^2 + y^2 = \rho^2$ into four segments, supplementary two by two. Describing upon these in each parallel plane the lunes of Hippocrates, we obtain four solid lunes, whose volume is proved to be equal to the volume of the common portion of the two intersecting cylinders, whose directrices are the meridian curves of the given surface of revolution, situated respectively in the planes (1) and (2), and whose corresponding generatrices are respectively parallel to the intersections of each parallel plane $z = k$ with (2) and (1). In the particular case of surfaces of revolution of the second degree, the volume of these lunes, between two parallel planes, does not involve any other irrationality than that which may be involved in the arbitrary distance $z_1 - z_0$ between the bounding planes. Thus, the expression for the volume of a solid, bounded by portions of a surface of revolution is free from the irrationality π.

In particular, the total volume of the lunes upon a sphere of radius a equals

$$\tfrac{1}{2} \cdot 8 \sin \theta \int_0^a (a^2 - z^2) dz = \tfrac{8}{3} a^3 \sin \theta,$$

θ being the angle of inclination of the meridional planes (1) and (2). For an hyperboloid of revolution of one sheet, where the meridian curve is an equilateral hyperbola, the volume of the lunes between the planes $z = z_0$ and $z = z_1$ equals

$$4 \sin \theta \int_{z_0}^{z_1} (a^2 + z^2) dz.$$

The analogy between these solid lunes and the circular lunes of Hippocrates is evident. All other ellipsoidal and hyperboloidal lunes are easily reduced to these. Paraboloidal lunes are also considered.

14. The connex (2, 2) of the ternary domain $a_x^2 u_\alpha^2 = 0$ coördinates to a line conic u_γ^2, the line conic $a_\gamma^2 u_\alpha^2$; to a point conic c_x^2, the point conic $c_\alpha^2 a_x^2$. If point and line conics be considered as flat spaces and points respectively in an S_5, the above coördination is expressed analytically as a connex (1, 1) in S_5. To a property of the connex (1, 1) in S_5 invariant under the projective group G_{35} of S_5 will correspond a property of the connex (2, 2) in S_2 invariant under the group G_8 of the plane which appears in S_5 as a subgroup of G_{35}.

Hence the invariant theory of the ternary connex (2, 2) breaks into two parts, one part invariant only under the projective group of the plane, the other invariant under a larger group. This second part may be very advantageously treated by reference to an S_5. Clebsch's (Lindemann) treatment of the connex (1, 1) in S_2 may be followed out for the connex (1, 1) in S_5 and the results obtained readily translated to apply to the connex (2, 2) in S_2.

Some facts as to the first part also are readily proven by considering the projective group of the plane as a subgroup of G_{35} in S_5 which leaves invariant a certain 4-spread and 2-spread. *E. g.*, The necessary and sufficient condition that the two quartics $a_x^2 b_x^2 (a\beta y)^2$ and $u_\alpha^2 u_\beta^2 \ (abv)^2$ may be put respectively in the forms

$$\sum_1^6 (c_x^2)^2 \quad \text{and} \quad \sum_1^6 (u_\gamma^2)^2,$$

where the six conics c_x^2 touch the line v and the six conics u_γ^2 go through the point y, is that $(abv)^2 (a\beta y)^2 = 0$.

15. Veronese, Segre and others have considered what may be termed the geometry of conics in which the conic defined

by six homogeneous coördinates $r_{11} : r_{22} : r_{33} : r_{23} : r_{13} : r_{12}$ is regarded as the element. Dr. Kasner's first paper is a contribution to this aspect of geometry. A linear equation in the coordinates $r_{\iota\kappa}$ defines a linear system of conics whose theory is well known. The general quadratic equation defines the quadratic system Q which is studied in detail by the author. It consists of ∞^4 conics, two of which pass through an arbitrary set of four points. The discussion of the degenerate conics contained in the system leads first to a correspondence defined by a quadri-quadric form f in two sets of line coördinates, and second to a curve of fourth class f, which is the envelope of the double lines contained in Q. The quartic covariant curve f is entirely general; to any quartic there correspond ∞^5 quadratic systems Q. The conics common to Q and one, two, or three linear systems are discussed. Finally analogous results are obtained for a quadratic system of quadric surfaces.

17. Peano has shown that a continuous curve

$$x = f(t), \quad y = \phi(t), \quad 0 \leqq t \leqq 1,$$

may fill a two dimensional region, in particular, the interior of a square. Peano's curve, however, has multiple points. If such points are excluded, then a curve defined by the above equations is called a Jordan curve. The question presents itself: May the external area of a Jordan curve have a positive value? This question is answered in the affirmative in Professor Osgood's second paper by means of an example. The paper has been published in the January number of the *Transactions*.

18. Mr. Young's paper is a continuation of the paper presented by him to the Society at its last summer meeting. He considers the theta-fuchsian functions of Poincaré associated with the group Γ of character $\{0, 3; 2, 4, \infty\}$ (Fricke-Klein notation), which has for fundamental circle the real axis. Every theta-fuchsian function of Γ of order $4\kappa + \alpha$ which does not become infinite at any point in the positive half-plane is the product of κ such theta-fuchsian functions of order 4 and one of order α, and has just κ movable zeros in a fundamental region C of Γ, unless $\alpha = 1$; in the latter case, it is the product of $\kappa - 1$ functions of order 4 and two of orders 2 and 3 respectively, and has just $\kappa - 1$ movable zeros in C. Theta-

fuchsian functions of any order exist with the proper number of arbitrarily assigned movable zeros. It is shown, further, how every such function can be rationally expressed in terms of two functions θ and ϕ, which are simple rational functions of two particular hyperelliptic theta constants. The functions θ and ϕ each vanish in a single point of C, and are such that θ^2 and $\theta\phi$ are theta-fuchsian functions of Γ of orders 2 and 3 respectively.

20. Scheffers has classified all hypercomplex number systems into quaternion or non-quaternion groups. Dr. Hawkes's paper gave a method for enumerating all types of non-quaternion systems of order n from types of order $n - 1$. These types follow the usual principles of classification in being inequivalent, irreducible, non-reciprocal, and with moduli. The method deduced is founded on certain theorems of Benjamin Peirce as extended by Dr. Hawkes in the *Transactions*, volume 3.

21. Dr. Stecker's paper justifies certain forms of equations of doubly infinite systems of geodesic lines used in a paper presented a year ago, where such systems were projected conformally into systems of conics.

22. The problem discussed in Dr. Kasner's second paper may be stated as follows: Given a doubly infinite system of plane curves $F(x, y, \lambda, \mu) = 0$, to investigate the existence and properties of those surfaces which can be represented point by point upon the plane in such a manner that the geodesics of the surface are represented by the assigned curves. Beltrami solved the problem for the straight lines, the corresponding surfaces being those of constant curvature. Dr. Stecker (volumes 2 and 3 of the *Transactions*) has considered the case of a linear system

$$\lambda\phi(x, y) + \mu\psi(x, y) + \chi(x, y) = 0,$$

imposing the additional requirement that the representation should be conformal. This case, however, may be reduced to Beltrami's; the surfaces are simply those of constant curvature, and the linear system, for conformal representation, must be conformally equivalent to a linear system of circles.

In the case of an arbitrary system $F = 0$ no solution in general exists. The necessary and sufficient conditions depend upon the consistency of a set of four partial differential equations in three unknown functions. The discussion of the number of solutions is carried out by means of Dini's theorem, the surfaces of Liouville type playing an exceptional rôle. Among the important results obtained are the two following: The only surfaces whose geodesics are capable of representation by circles are those of constant curvature. The only surfaces whose geodesics can be conformally represented by curves of the form $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ are the developable surfaces and those whose linear element may be written $ds^2 = v(du^2 + dv^2)$. The paper will appear in the *Transactions.*

23. Many properties of Laplace's equation in two dimensions can be deduced from the theory of functions of a complex variable. In this way the two following theorems may be easily proved, in each of which we suppose the function $u(x, y)$ to be harmonic throughout the neighborhood of a point P: 1° If u becomes infinite for *no* method of approaching P, it will, if properly defined at P, be harmonic there. 2° If u becomes infinite for *every* method of approaching P, it has the form $C \log r + v$ where C is a constant, r the distance from P, and v a function which is harmonic at P. According to a remark of Professor Osgood, the first of these theorems is an immediate consequence of the second. In the present paper Professor Bôcher shows how the second theorem (and therefore also the first) can be extended to cases to which the original method of proof was in no wise applicable — on the one hand to Laplace's equation in three dimensions, and on the other hand to the general linear partial differential equation of the second order of the elliptic type in two independent variables and with analytic coefficients. The method of proof consists in an application of Green's theorem.

24. In the paper of Mr. Birkhoff and Mr. Vandiver the complement of two well-known arithmetical congruence theories is considered. The two theories (given in most text-books) are: 1° Theory of power residues; 2° Theory of the solution of binomial congruences. But the relation $A^n - 1 \equiv 0 \pmod{p}$, or more generally $a^n - b^n \equiv 0 \pmod{p}$, where A, a, b and p are integers, may be considered from another standpoint; given a, b and n, we may ask the question: What are the properties of the divisors of $a^n - b^n$, or will certain classes of

divisors always exist? The theory arising from efforts to solve these problems may be regarded as a complement to the two theories just mentioned, and forms the subject of the present discussion.

A primitive divisor of $a^n - b^n$ is defined as a divisor which is prime to all the factors of the form $V_k = a^k - b^k$, k being less than n. Then, using several theorems due to Euler and Sylvester (which are also proved), the following theorem (which we believe new) is arrived at: If $n > 2$, then every integral form $a^n - b^n$ possesses at least one primitive divisor, with the single exception $2^6 - 1$.

The paper concludes with applications of the theory of cyclotomic divisors to the "instantaneous" demonstration of the two theorems: 1° If n is any integer, there is an indefinite number of primes of the form 1 (mod n). 2° The equation $(x^p - 1)/(x - 1) = 0$ is irreducible if p is prime.

25. The main steps in the proof given in Professor Miller's paper are as follows: It is first observed that the continued product of all the operators of an abelian group G is the identity whenever G contains more than one operator of order 2. If G contains only one operator S of order two, then S is always the continued product of all the operators of G. The group of isomorphisms I of a cyclic group C must contain more than one operator of order 2 unless the order of C is one of the three numbers 4, p^a, $2p^a$, where p represents any odd prime. Without employing the theory of congruences it is proved that in these three special cases I contains just one operator of order 2. The generalized Wilson's theorem follows directly from these facts. The paper has been offered to the *Annals of Mathematics* for publication.

26. In Professor Morley's paper the factors of the determinant $|(x_i - a_j)^{-2}|$ are found by a process which appears to be novel; and thereby the geometric theorem is proved that, given a rational norm curve R_n through $n + 1$ points in S_n, a quadric apolar to the $n + 1$ points cuts R_n in points which form a system conjugate with the $n + 1$ spaces S_{n-1} determined by the $n + 1$ points.

F. N. COLE.

COLUMBIA UNIVERSITY.