Since $S_1T = c^{-1}b^{-1}cbe$ is the transform of $c$ by $bc$, it is of period three.

The final relation (10) becomes

\[
(b^{-1}b^{-1}c \cdot b^{-1}cbe)^2 = (c^{-1}bcb^{-1} \cdot b^{-1}cbe)^2 = (c^{-1}bcb^{-1}cbe)^2
\]

\[
= e^{-1}b(eb^2)^4b^{-1}c = I.
\]

Since $S_1$ is commutative with $S_1''$, the condition $S_3 = I$ follows from $(b^{-1}c^{-1}b^2c^{-1})^3 = I$ or $(eb^2c)^3 = I$.

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NOTE ON A PROPERTY OF THE CONIC SECTIONS.

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It is easily proved that if $P, Q, R$ are any three points on the conic $Ax^2 + By^2 = 1$, and $O$ the center of the conic, then the areas of the triangles $OPQ$, $OPR$, $OQR$ will satisfy an equation independent of the position of the points $P, Q, R$.

If $a, b, c$ are the areas in question, this equation is

\[(1)\quad a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.
\]

Now we can prove that such an invariant relation is possible for no plane curves except the central conies; i.e., if we seek a plane curve $C$ and a point $O$ in its plane such that, if $P, Q, R$ are any three points on $C$, the triangles $OQR$, $ORP$, $OPQ$ are connected by a relation independent of the coordinates of the points $P, Q, R$, we find $C$ to be a central conic section and $O$ its center.

To prove this theorem, let $O$ be the origin of coordinates, and let the coordinates of $P, Q, R$ be respectively $x_1, y_1; x_2, y_2; x_3, y_3$. Then twice the areas of the three triangles are

\[
2a = \pm (y_2x_3 - y_3x_2), \quad 2b = \pm (y_3x_1 - y_1x_3),
\]

\[
2c = \pm (y_1x_2 - y_2x_1),
\]
which expressions are functions of the three independent variables \( x_1, x_2, x_3 \); \( y \) being considered a given function of \( x \) for points on the curve.

As \( a, b, c \) must satisfy a relation independent of \( x_1, x_2, x_3 \), the Jacobian \( \partial(a, b, c)/\partial(x_1, x_2, x_3) \) must vanish. If \( y_1' \) represents \( dy_1/dx_1 \), etc., we find

\[
y_2'\{y_2[x_iy_2 - x_2y_1 + x_1x_2(y_2' - y_1')] + x_3(x_2y_1y_2' - x_1y_2y_3')
\]
\[
+ x_3[(x_1y_2 - x_2y_1)y_2' + y_1y_2(y_2' - y_1')] + y_3(x_2y_1y_2' - x_1y_2y_3') = 0,
\]
say

(2) \[ y_3'(y_3k + x_3l) + x_3m + y_3l = 0, \]

\( k, l, m \) being functions of \( x_1, x_2 \) only, and therefore independent of \( x_3 \).

Two cases (a) and (β) may now present themselves as follows:

(a) The functions \( k, l, m \) are not all identically zero. In this case the equation (2) gives, when integrated,

(3) \[ y_3^2k + 2y_3x_3l + x_3^2m = f(x_1, x_2). \]

If we give to \( x_1 \) and \( x_2 \) arbitrary constant values, the equation (3) represents a conic section with its center at \( O \).

(β) The functions \( k, l, m \) are all zero. We must then have \( x_2y_1' - y_2 = 0 \). Giving to \( y_1' \) a definite constant value, we obtain the equation of a straight line—a special case of (3).

The theorem stated above is therefore proved.

It may be noticed that \( f(x_1, x_2) \) in (3) may be multiple valued. The equation will then represent a series of similar conics similarly placed. If these are finite in number, say \( n \), we find that, if \( P, Q, R \) be located anywhere on this system of curves, the areas \( a, b, c \) of the three triangles considered will satisfy an equation of degree

\[ 6n + 18n(n - 1) + 6n(n - 1)(n - 2) \]

at most, whose left-hand member is composed of factors of form similar to (1), as the reader may prove without much difficulty.