CESÀRO'S INTRINSIC GEOMETRY.

Lezioni di Geometria Intrinseca. By ERNESTO CESÀRO, professor of mathematics at the Royal University of Naples. Published by the author, Naples, 1896. 8vo., 264 pp., 48 figures.

Ernesto Cesàro, Vorlesungen über natürliche Geometrie. Authorized German translation by Dr. GERHARD KOWALEWSKI, docent at the University of Leipzig. Teubner, 1901. 8vo., vi + 341 pp., 48 figures.

THE subject of intrinsic geometry is almost unknown among American mathematicians, yet it has quite an extensive literature, which has been recently collected and published by Wölffing.* The name Cesàro occurs more frequently than any other in this report.

The work here under consideration contains considerable new matter, but its principal mission is to provide a systematic treatment of the subject from the beginning. The success of the effort is attested to by the fact that the work received honorable mention at the awarding of the Lobachevsky prize in 1897.

The German translation differs but little from the Italian original, except that it corrects a considerable number of typographical and other errors, and embodies in the text a number of notes appended to the original. The further changes will be mentioned later.

The work is divided into seventeen chapters, of which eight treat of plane geometry, two of twisted curves, three of surfaces, one of line congruences, and finally the last three of three-dimensional space, of curves in hyperspace, and of hyperspaces, respectively. There is also an appendix giving a generalization of Grassmann's numbers, a discussion of the equilibrium of flexible but non-extensible wires, and the equations of elasticity in hyperspace.

The physical improvements in the German edition, the spaced paragraphing, the full index (wholly lacking in the original)

* Bericht über den gegenwärtigen Stand der Lehre von den natürlichen Coördinaten, *Bibliotheca Mathematica*, Ser. 3, vol. 1, pp. 142–159.

and the more distinctive type used for printing the illustrative examples, are all of great assistance to the reader.

The subject matter proper of the book is not preceded by an introduction, and no references are given, both of which omissions greatly limit the usefulness of the work. The lack has now to a large extent been supplied, so far as the plane is concerned, by Wölffing's report mentioned above.

In the first five chapters a curve is defined by a relation between s, the length of arc measured from any point of the curve and ρ, the radius of curvature. Frequently rectangular coördinates (u, v) referred to the tangent and normal are used. The first chapter gives a good sample of the methods employed, by deriving a large number of interesting properties of certain curves in a brief and strikingly elegant manner.

Let $M(o, o)$ and $M'(u, v)$ be two points on a curve. Since $\sqrt{u^2 + v^2} < \delta s < u + v$, and $\lim v/u = 0$, it follows that $\lim \delta s/u = 1$. If MM' makes an angle $\delta\phi$ with the tangent at M, $1/\rho = \lim \delta\phi/\delta s$, $\lim v/u^2 = 1/\rho$. These ideas suffice to obtain the forms of the circle $\rho = a$, the catenary $\rho = a + s^2/a$, and the catenary of uniform resistance $\rho = \frac{1}{2}a(e^{-s/a} + e^{-s/a})$. It is to be observed that the origin and axes are variable and are only intermediate steps in the analysis.

Inflexions and cusps appear according as

$$\lim \frac{v}{u^{2-n}} = \frac{1}{(2-n)(1-n)} \lim \frac{s^n}{\rho}$$

is less or greater than zero. Asymptotes are characterized by $s = \infty$ when $1/\rho = 0$ and $\lim \phi$ is finite. Asymptotic circles result when $\lim \rho = a$, as $\phi = \infty$, and asymptotic points appear when $a = 0$. The study of these points closes the chapter.

The examples discuss the involute of the circle $\rho^2 = 2as$, the form of the tractrix, the proof that the segment of the tangent cut off by the asymptote is constant, and a simple construction for the center of curvature of any point. The German edition compares the forms of the two catenaries, and shows that the catenary of uniform resistance has two parallel asymptotes. The cycloids are defined by the equation

$$\frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1\,;$$

the hypocycloid or epicycloid is defined, according as a is greater or less than b, and the common cycloid is the intermediate form, when $a = b$. The whole discussion, including eight figures, occupies less than two pages. Its directness must be admitted, but the periodic form of the curve is not established until an auxiliary variable is employed. From the form of the equation, no direct knowledge of the form of the curve beyond a cusp is given. The author makes no further mention of the matter, but ignores the fact that he departs from his otherwise consistent procedure to establish the result. This departure is, however, unnecessary if account be taken of the change of sign which the arc undergoes when passing a cusp. The same scheme will apply to all the curves which have cusps, whether periodic or not. This excellent list of examples closes with a discussion of the curves

$$\rho = ks^n,$$

which include the involute of the circle ($n = \frac{1}{2}$), the logarithmic spiral ($n = 1$) and the double spiral ($n = 2$) which Cesàro calls the clotoid.*

The second chapter discusses the fundamental formulas

$$\frac{dx}{ds} = \frac{y}{\rho} - 1, \quad \frac{dy}{ds} = -\frac{x}{\rho},$$

which are necessary and sufficient for the immovability of the point (x, y) not on the curve. These formulas are all that are needed for expressing the equation of any locus which is invariantly connected with the curve. Here envelopes, evolutes, parallel curves are discussed and a rich collection of examples is given which illustrate how to derive the intrinsic equations of curves defined as a geometric locus. The method is rather too abstract, and not sufficient use is made of coördinate axes. For example, it is proved that the evolute of a cycloid is a similar cycloid, but an indirect procedure is necessary to determine the relative positions of the two curves. The same fact is still more evident in case of the hypocycloid and epi-

* For a further discussion of the clotoid, and the literature upon it, see Loria's recent work, Le curve piane algebriche e transcendenti, teoria e storia; German by Schütte, Specielle algebraische und transcendente Curven in der Ebene, Theorie und Geschichte, Leipzig, Teubner, 1901, p. 457. It would be interesting to compare the various double spirals with the stereographic projection of the logarithmic spiral.

cycloid. The relation between ρ_n, the radius of curvature of the nth evolute, is shown to satisfy the equation

$$\rho_n = \rho \frac{d\rho_{n-1}}{ds}.$$

The third chapter is devoted to the discussion of particular curves. It begins with a review of five pages on the ordinary properties of conics, derived by the well known methods, after which the equation between s and ρ is expressed by means of an elliptic integral of the second kind. The equation of the cassinian oval is expressed as a hyperelliptic integral of the first kind, for which $p = 3$. A family of curves is also discussed for which Wölffing has suggested the name Cesàro curves. They are defined by the following property: the radius of curvature at any point is proportional to the segment on the normal between the point of tangency and its intersection with the polar of the point of tangency with regard to a fixed circle. When the circle reduces to a point the curve becomes a sinuous spiral. When the circle becomes a straight line, the locus is called a Ribaucour curve. Although various curious properties are derived, the method does not seem adapted to the study of the general topology of the curves; no figures are given except of the well known forms. The German edition has added one more paragraph [§ 46] to this chapter, dealing with the investigation of those curves whose osculating circles, dilated from the point of contact, all belong to a linear complex of circles. The locus of centers is a Cesàro curve.

In the fourth and fifth chapters, contact and osculation, and roulettes, the power of the method again becomes apparent. The ordinary properties of contact, curvature, successive evolutes, maximum and minimum curvature, etc., are very skilfully derived. In the chapter on roulettes a more detailed study of the cycloid is given, and a number of elegant relations between various Cesàro curves are derived.

Thus far the procedure has been successive; a small number of formulas was first derived and a large number of consequences drawn from them. The remaining chapters (except the ninth and tenth) use many auxiliary coördinates.

Chapter VI deals with barycentric coördinates. It would grace any work on modern geometry, and the analysis seems to lend itself readily to the method in question. Centers and lines of gravity are discussed at length and applied to a large number of curves. Chapter VII, barycentric analysis, is a direct continuation of the preceding. Equations of a line, pair of lines, conic section, tangent, normal, pole and polar, are derived in much the same manner as in the ordinary texts on trilinear coördinates. The chapter closes with a well written paragraph on the symmetric triangular curves and the anharmonic curves. The lack of references here seems particularly unfortunate, for it is of considerable value to compare the treatment here given with that of Laguerre, Halphen, Klein and Lie.

The last chapter in the part on plane geometry is devoted to pencils of plane curves. Let u be a function of position in the plane. If u changes by the amount du, when the point moves through the distance ds,

$$\frac{du}{ds} = \cos\omega \frac{\partial u}{\partial s_1} + \sin\omega \frac{\partial u}{\partial s_2},$$

in which ds_1, ds_2, represent orthogonal components of ds. The expression

$$\Delta u = \left(\frac{\partial u}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_2}\right)^2$$

is called the first differential parameter. If the pencil $u \equiv \kappa$ consists of parallel curves, Δu is a function of u alone. The second differential parameter

$$\Delta^2 u = \left(\frac{\partial}{\partial s_1} + \frac{\partial \omega}{\partial s_2}\right)\frac{\partial u}{\partial s_1} + \left(\frac{\partial}{\partial s_2} + \frac{\partial \omega}{\partial s_1}\right)\frac{\partial u}{\partial s_2}$$

vanishes for an isothermal system, provided u is the isometric parameter. These considerations furnish a good introduction to curvilinear coördinates, their fundamental formulas, and the conditions for integrability which follow. Then are added the relations among curvilinear coördinates, which were first derived by Lamé and by Bonnet, but which are usually found in works on the theory of surfaces.

The remainder of the book is devoted to geometry of more than two dimensions. Chapter IX treats of twisted curves and ruled surfaces. In the first part pure intrinsic geometry again becomes prominent, but the digression on the straight line and the application of the results to scrolls, developables, and associated surfaces makes a wide departure from the otherwise consistent development. One has the feeling that the arrangement is far from natural in the sense of each result suggesting the following one, and the deductions frequently leave several steps unnoticed or assumed. The proof of the theorem that a developable must be the envelope of tangents of a twisted curve is direct and elegant. In establishing Chasles's correlation between tangent planes and points of tangency on a scroll the two lines g, $g + \delta g$ are confused; the tangent plane should contain the generator through the point of tangency. The next chapter discusses some particular twisted curves. Spherical curves, helices, and conical loxodromes are treated at some length and a number of miscellaneous examples are added. A discussion of the Bertrand curves, defined by a linear relation between curvature and torsion, is followed by a study of the curves satisfying the quadratic relation

$$\frac{A}{\rho^2} + \frac{B}{r^2} + \frac{C}{\rho r} = \frac{P}{\rho} + \frac{Q}{r}.$$

Although the various properties mentioned are all derived, the method of deduction is so briefly outlined and explanations are so sparingly used that the text would prove to be almost impossible reading to any one not fairly familiar with the results of the ordinay theory.

Chapter XI, general theory of surfaces, is largely geometric in character and not so materially different from the treatment in the ordinary books, except that it is remarkably brief and direct. Most of us are not satisfied with these considerations as proofs, but they are of immense value when accompanied by the lively geometric interpretation. There are no natural coördinates here as in the case of curves, but both this treatment and the ordinary one would be greatly improved by drawing from each other. One interesting theorem is the determination of the curvature in the tangent plane in a hyperbolic point (theorem of Beltrami), as the theorems of Euler and Meusnier do not apply to this case. The result shows that the curvature of the section made by the tangent plane is two thirds that of the asymptotic line which it touches.

The next statement seems trivial; it says that two curves may touch each other and have the same osculating plane at the point of contact without having the same curvature at that point.

Clairaut's theorem regarding geodesics on surfaces of revolution is very neatly proved.

The German edition adds a theorem regarding the curvature of a normal section of the circumscribing cylinder of a surface (curve of apparent contour).

A good discussion of surfaces of constant curvature is given, and figures of the various types are discussed. Asymptotic lines on the pseudosphere are derived and the form of the lines obtained. Applicability, minimum surfaces and the Weingarten surfaces are briefly treated.

These two chapters are very valuable when read in connection with the older treatment of the subject; but the outlook is by no means hopeful that the later method will prove sufficiently powerful to secure many new results from the theory of surfaces. The chapter closes with rather a full discussion of surfaces of the second order.

Chapter XIII, infinitesimal deformations of a surface, discusses the possible deformations and the consequent changes of curvature. It is skilfully proved that the mean curvature of a mimimum surface must be everywhere zero, and that the curvature of any surface is not changed by deformation.

The next chapter is devoted to rectilinear congruences. It is a rapid review of the work of Kummer, which is so well presented in the work of Bianchi. A certain arbitrariness is here apparent and a strong suspicion is unavoidable that all the results were known beforehand; moreover, one feels that the present method is forced somewhat in reproducing those results. The results of Sturm on the configuration of the normals of a surface near a given point are added to the older theorems of Hamilton and Kummer, and some new relations among the derivatives are obtained analogous to those of Codazzi for surfaces. By means of these there is given a noteworthy proof of the theorem that the mean envelope of an isotropic congruence is a minimum surface. The second half of this chapter is considerably altered in the German edition.

Chapter XV treats of three dimensional space. The discussion is very similar to the introduction of curvilinear coördinates in the plane. All the tangent lines to a surface at a particular point lie in a plane. In any triply orthogonal system the curves of intersection are lines of curvature. Every pencil of parallel surfaces belongs to a triple orthogonal system. Several paragraphs are devoted to the derivation of formulas analogous to Codazzi's.

Thus far the space is considered linear, so that the element of arc, referred to the tangents of the three orthogonal lines of curvature, is expressed in the form

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

If the space is not linear but has a constant curvature $1/R$, it is shown that the element of arc is defined by

$$ds = \frac{\sqrt{dq_1^2 + dq_2^2 + dq_3^2}}{1 + \frac{1}{4R^2}(q_1^2 + q_2^2 + q_3^2)}.$$

Apart from giving the impression of being rather overloaded, this chapter is very interesting and the power of the methods is made particularly apparent in all of the geometric part of it.

Chapter XVI treats of curves in space of n dimensions. The various normals are first defined, the conditions for orthogonality deduced, and then the analagons to Frenet's formulas are derived, which are called the fundamental formulas for the intrinsic analysis of curves in hyperspace.

In the German edition an article is added which is interesting because it discovers a fundamental difference in the nature of spaces having an even number of dimensions from those having an odd number. The locus of a point having first, second, $\cdots$, nth curvatures all constant is spherical or helical, according as n is even or odd.

The last few pages of this chapter are devoted to barycentric analysis of hyperspace, quite analogous to the projective system of Professor Stringham.

The last chapter preceding the appendix is on hyperspaces. The generalizations of the theorems of Euler and Dupin are first derived, then the formulas of Codazzi and the criteria for infinitesimal deformation. The volume closes with the remarkable theorem of Beez that a non-extensible hypersurface cannot be deformed.

The book represents an important and a beautiful work, and the reader's admiration is constantly excited for the author's versatility and power of assimilating the existing literature.

Two general defects seem to be patent, one in presentation, one in range. It is the author's aim to emphasize the force and directness of the analysis by having the results appear after a minimum of intermediate work, but this work has frequently been so abbreviated that the reading is at times extremely difficult. Again, the author was naturally ambitious to have his method apply to as large a range of subjects as possible, but its applicability seems but questionably successful when applied to projective properties.

For the differential analysis of curves and surfaces the work of Cesàro is certainly a powerful instrument and can be used with profit by every student of the subject.

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CORNELL UNIVERSITY,
December, 1902.

GAUSS'S COLLECTED WORKS.

Carl Friedrich Gauss Werke. Achter Band. Herausgegeben von der königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B. G. Teubner in Leipzig, 1900. 4to., 458 pp.

WITH the death of Ernst Schering, the venerable editor of the first six volumes of Gauss's collected works, it became incumbent on younger hands to continue the labor of studying, selecting and preparing for the press everything of interest that still remained in the great mass of manuscript that Gauss left behind. Professor Klein, with customary energy and dispatch, has made the necessary arrangements to bring Schering's labor to a prompt termination.

Four new volumes are planned. Professor Brendel, of Göttingen, is editor-in-chief, having the editorial supervision of the entire undertaking. He has also, in particular, the preparation of volume VII. This is to contain the final edition of the Theoria motus and Gauss's voluminous work on the theory of perturbation of the smaller planets, the theory of the moon, etc.

Volume VIII is the volume under review and will be spoken of later.

The contents of volume IX fall into two parts. One of these is to be devoted to mathematical physics, and is in charge of Professor Wiechert, director of the magnetic observatory in