THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 28, 1903. The following thirty-one members of the Society were present at the morning and afternoon sessions:

Dr. Grace Andrews, Professor Joseph Bowden, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Professor Achsah M. Ely, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Dr. E. R. Hedrick, Professor L. I. Hewes, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Dr. Emory McClintock, Professor James Maclay, Professor H. P. Manning, Professor T. F. Nichols, Professor W. F. Osgood, Miss I. M. Schottenfels, Professor D. E. Smith, Professor P. F. Smith, Mr. C. E. Stromquist, Professor H. D. Thompson, Miss Mary Underhill, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Mr. H. E. Webb, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair, being relieved by Vice-President Professor W. F. Osgood. The Council announced the election of the following persons to membership in the Society: Professor F. H. Bailey, Massachusetts Institute of Technology; Mr. A. T. Bell, High School, Reynolds, Ill.; Professor F. P. Brackett, Pomona College, Claremont, Cal.; Mr. W. E. Breckenridge, Morris High School, New York, N. Y.; Professor Ellen L. Burrell, Wellesley College; Miss E. B. Cowley, Vassar College; Professor E. E. De Cou, University of Oregon; Mr. F. D. Frazer, University of Oregon; Professor J. Willard Gibbs, Yale University; Dr. C. N. Haskins, Massachusetts Institute of Technology; Mr. A. C. Lunn, University of Chicago; Mr. C. L. E. Moore, Cornell University; Mr. F. G. Reynolds, College of the City of New York; Mr. C. E. Stromquist, Yale University; Professor W. E. Taylor, Syracuse University; Mr. Charles Van Orstrand, U. S. Geological Survey, Washington, D. C. Five applications for admission to the Society were received.

Professor E. W. Brown was reëlected a member of the Editorial Board of the *Transactions* for a term of three years. The office of Assistant Secretary of the Society, vacated by the appointment of Dr. Edward Kasner to the editorial staff of the *Transactions*, was abolished.

The following papers were read at this meeting:

(1) Dr. L. P. EISENHART: "Congruences of curves."

(2) Dr. EMORY McCLINTOCK: "The logarithm as a direct function."

(3) Professor H. P. MANNING: "Non-euclidean geometry of nets of circles."

(4) Mr. C. E. STROMQUIST: "A generalization of the length integral."

(5) Dr. EDWARD KASNER: "Three notes on projective geometry."

(6) Mr. W. B. FORD: "A theorem concerning the functions of two or more complex variables."

(7) Professor W. F. OSGOOD: "The integral as the limit of a sum, and a theorem of Duhamel."

(8) Dr. E. R. HEDRICK: "The integral curves of a partial differential equation."

(9) Professor E. B. VAN VLECK: "On an extension of the 1894 memoir of Stieltjes."

(10) Dr. A. S. GALE: "On a generalization of the set of associated minimal surfaces."

(11) Professor G. A. MILLER: "A fundamental theorem with respect to transitive substitution groups."

(12) Professor E. W. BROWN: "On the derivatives of the lunar coördinates with respect to the elements."

(13) Professor CHARLOTTE A. SCOTT: "On the fundamental theorem of projective geometry."

(14) Professor ALFRED LOEWY: "Ueber die Reducibilität der reellen Gruppen linearer homogener Substitutionen."

Professor Loewy's paper was communicated to the Society through Professor E. W. Brown. In the absence of the authors, the papers of Mr. Ford, Professor Miller, Professor Brown, Professor Scott, and Professor Loewy were read by title.

The papers of Dr. McClintock and Professor Miller will appear in the BULLETIN. Professor Scott's paper is published in the April number of the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Eisenhart considers congruences of curves, the coördinates of whose points are given by the equations

$$x = f_1(t, u, v), \quad y = f_2(t, u, v), \quad z = f_3(t, u, v),$$

where u and v denote the parameters determining the curves and t is the parameter which determines points on the curve. After establishing several theorems which Darboux found from another point of view, and giving a few examples in illustration, the discussion proceeds to the problem of finding a function $\phi(u, v)$ such that the tangents to the curves of the congruence at the points of intersection with the surface defined by the above formulæ after t is replaced by ϕ, shall form a normal congruence, or in the second place, the ruled surfaces $u =$ const., $v =$ const. of this congruence of tangents shall be developable. For every congruence a function ϕ exists which furnishes a solution of the former of these problems, but not so for the second problem. However, when the curves $t =$ const., $u =$ const. on the surfaces defined by the above equations when v is constant, and when the curves $t =$ const., $v =$ const. on the surfaces $u =$ const., form a conjugate system on each, the function ϕ can be a constant, and only in this case.

When a relation between the parameters, such as $v = \phi(u)$, is given the corresponding curves form a surface and in some cases this function can be so chosen that all the tangents to these curves form a normal congruence. It is evident that the tangents to all the curves of the congruence form a complex, and the determination of congruences such that this complex be linear requires the integration of a linear differential equation of the first order.

The above expressions for x, y, z define space referred to a triple system of surfaces $t =$ const., $u =$ const., $v =$ const., and in the preceding discussion they have been considered as defining the congruence C_t of the curves of intersection of the surfaces $u =$ const., $v =$ const. It is evident however that these equations may equally well be considered as defining the congruences C_u and C_v composed of the intersections of the surfaces $t =$ const., $v =$ const. and $t =$ const., $u =$ const. respectively. On this account these equations may be said to define a *triple congruence* of curves. The theorems of the preceding discussion when viewed in this threefold light lead to new results. The existence of a triply rectilinear congruence is established and a few of its properties found. The paper closes with a consideration of the three complexes of tangents to the curves of a triple congruence.

3. Professor Manning's paper is in abstract as follows: We can construct a geometry by taking for straight lines the circles of a net. There are three cases according as the radical center of the net is outside, on, or within all the circles. These three cases give us the hyperbolic, parabolic, and elliptic geometries, respectively. Considering only the first case, we find a real circle which cuts orthogonally all the circles of the net and which may be called the absolute. Distance between two points is proportional to the logarithm of the cross ratio of these points taken with the two points where the line joining them meets the absolute. Circles which do not meet the absolute are circles in this geometry, circles tangent to the absolute are boundary curves, and circles intersecting the absolute are straight lines or equidistant curves. Bilinear substitutions which have the absolute for invariant circle represent the three kinds of motion possible in this geometry: translation along a line and a system of equidistant curves, translation along a system of boundary curves, and rotation. Of particular interest is the group of the rotations which leave invariant a system of alternately symmetric and congruent triangles covering the plane.

4. In his dissertation (Göttingen, 1901), Georg Hamel considered those geometries in which the straight lines are the shortest distances, and determined the most general length integral for which this is the case. His work was based on the axioms laid down by Hilbert in his Grundlagen der Geometrie, from which he derived an expression for the length as a definite integral.

In Mr. Stromquist's paper those geometries are considered for which the circles whose centers are on the x-axis are the shortest distances. And the most general length integral (with certain restrictions which are laid down) for which this is the case is obtained in the form

$$\int_{x_1}^{x_2}\left[y^2\int_c^{y'}\int_c^{y'}W\{y^2(1+y'^2),\, x+yy'\}\,dy'\,dy' + yy'\int_0 W(y^2,x)\,dx + \frac{d\phi(x,y)}{dx}\right]dx,$$

where W, regarded as a function of x, y, and y' $(= dy/dx)$, is *any* continuous function satisfying the condition that $W \gtreqless 0$ according as $-\pi/2 \lesseqgtr \theta(=\tan^{-1} y') \lesseqgtr +\pi/2$; and where ϕ is an arbitrary function of x and y.

This expression is then further specialized under the assumption that length $\overline{AB} =$ length $\overline{BA}$, where the length is measured along the same curve.

The condition that extremals should be perpendicular to the transversals (*i. e.*, that "radii" are perpendicular to "circles") is then discussed in general. In particular it is shown: 1° that the geometry in which straight lines are the shortest distances reduces to the euclidean geometry; 2° that the geometry in which the above circles are the shortest distances reduces to the automorphic figure (*i. e.*, the figure obtained by a conformal transformation of the pseudosphere on the plane); and 3° that, in general, any geometry in which extremals are perpendicular to the transversals may be obtained by a conformal transformation of some surface on the plane. The converse of this theorem is obviously true by Gauss's theorem.

5. Dr. Kasner's first note is entitled "A relation between projective and circular transformations." Consider a general direct circular transformation (linear transformation of the complex variable) Γ and a finite point O. The bundle of circles through O is transformed by Γ into the bundle of circles through the corresponding point O'. There is thus established a correspondence between the centers of corresponding circles. This correspondence is proved to be homographic. The projective transformation H obtained in this way depends upon both Γ and O. It is then proved that H is entirely general, *i. e.*, given any H it is possible to find a circular transformation Γ and a point O which together give rise to H. The relation between Γ, O on the one hand, and H on the other is unique, provided O is finite and not the pole of Γ, and H does not transform the line at infinity into itself. The result for the indirect circular transformations (linear transformation of the conjugate complex variable) is analogous. The relation obtained is applied to the formation of invariants of systems of circles by means of the projective invariants of their centers. Finally the relation is extended to space, but the projective transformation is no longer of the most general type.

The second note, "On the projective geometry of the plane pentagon," deals with certain pentagons covariantly related to a given pentagon. If the vertices of the original pentagon P are a, b, c, d, e, then the *diagonal* pentagon P' is obtained by drawing the diagonals ac, bd, ce, da, eb; and the *inscribed* pentagon P_1 is obtained by inscribing in P a conic and connecting the points of contact. The consideration of certain polarities proves Clebsch's theorem that P and P' are homographic, and also that P and P_1 are homographic. The two homographies obtained are commutative, hence the pentagon inscribed in P' coincides with the diagonal pentagon of P_1. Repetition of the diagonal construction gives a series of pentagons $P, P', P'', \cdots$ which define (at least for a convex pentagon) a definite limiting point investigated by Clebsch. The same point is obtained by indefinite repetition of the construction for the inscribed pentagon. The common limiting point may be constructed as one of the fixed points of a collineation, or as one vertex of the common self-conjugate triangle of certain conics. Corresponding points for polygons of more than five sides exist, but their theory seems to be much more complicated, if not transcendental.

The third note will appear in the BULLETIN as a separate paper.

6. In § 62 of Dini's work, Serie di Fourier, etc., the author establishes certain results related to the theory of functions of one complex variable, of great utility, as he subsequently shows, in the study of the convergence of infinite series. The purpose of Mr. Ford's paper is to generalize the results thus established for functions of one complex variable to the case where functions of two or more complex variables may appear. Thus a theorem is obtained which, it is believed, has important relations to the study of the convergence of a multiple series, just as Dini's results are useful in the study of similar questions pertaining to a simple series.

7. In the application of the integral calculus to problems in physics and mechanics the theorem that in the limit of a sum any infinitesimal may be replaced by another infinitesimal that differs from it by one of higher order is fundamental. The formulation which Duhamel gave for this theorem has been challenged by Mansion, who shows that it may be so understood as to lead to false results; but Mansion does not suggest a correct formulation applicable to the cases that arise in practice. The difficulties are similar to those encountered in integrating

an infinite series term by term and may be met by a requirement analogous to that of uniform convergence. The object of this paper is to give a rigorous formulation of Duhamel's theorem, and at the same time one which is adapted to the applications of the theorem which present themselves in practice.

8. Dr. Hedrick's paper forms an extension of that presented at the Christmas meeting 1901, on the characteristics of differential equations. It is found that the methods employed to obtain the characteristic lines may be developed so as to lead to a new and more rigorous treatment of integral curves. The paper is then devoted to an investigation of the cases in which the ordinary Cauchy-Kowalewski existence theorem breaks down, when applied to a general space curve.

9. The paper of Professor Van Vleck treated an extension of a memoir published by Stieltjes in two parts in the *Annales de Toulouse*, 1894–95. Stieltjes there gives an exhaustive and elegant investigation of a continued fraction

$$(1) \qquad \frac{1}{a_1 z +}\;\frac{1}{a_2 +}\;\frac{1}{a_3 z +}\;\frac{1}{a_4 +}\;\cdots$$

in which the coefficients a_n are positive. This he connects, on the one hand, with a series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \cdots$$

whose coefficients are conditioned by the relations

$$A_n \equiv \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} > 0, \quad B_n \equiv \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix} > 0,$$

and, on the other hand, with one or more functional integrals

$$\Phi(z) = \int_0^\infty \frac{df(u)}{z+u}$$

in which $f(u)$ is a monotone function of u. In his extension Professor Van Vleck imposes only the condition that $a_{2n} > 0$ or that $a_{2n+1} > 0$. The corresponding restriction upon the series is that $B_n > 0$, $A_n \neq 0$ or $A_n > 0$, $B_n \neq 0$. If the even and odd convergents are considered apart, instead of being combined by (1) into a single continued fraction, the subsidiary conditions, $A_n \neq 0$ and $B_n \neq 0$ respectively, may be dispensed with. The integrals which give rise to the continued fraction or series have the form

$$\int_b^a \frac{df(u)}{z+u} \quad (-\infty \leqq b < a \leqq +\infty).$$

The condition imposed by Stieltjes upon the continued fraction is doubtless the simplest one possible. It not only suffices to ensure the reality of the roots of the numerators and denominators of the convergents, the roots lying upon the negative half of the axis of z, but it also necessitates certain theorems concerning the alternation of the roots, when two convergents are compared. These theorems, which form the basis of the investigation of Stieltjes, hold also for the more extended class of continued fractions discussed in the present paper, provided that the positive and negative half axes are considered separately. The greatest difference between the theory of Stieltjes and the extended theory appears in the consideration of questions of convergence. One new feature was a divergent series which represented in the positive and negative half planes two distinct analytic functions having the real axis as a natural boundary.

10. It is well known that a minimal surface is a surface of translation whose generators are its minimum lines. Dr. Gale considered surfaces of translation whose generators are any imaginary curves defined by functions of conjugate complex parameters. These surfaces may also be regarded as surfaces whose coördinates satisfy Laplace's equation and may therefore be legitimately spoken of as harmonic surfaces. The set of harmonic surfaces discussed is given by the equations of the set of associated minimal surfaces, omitting the conditions that the generators be minimum curves. These surfaces are intimately related to the "associated surfaces of negative curvature" which Professor Maclay discussed at the meeting of the Society on February 22, 1902. The paper will appear in the *Annals of Mathematics*.

12. The derivatives of the lunar coördinates are easily obtained when the literal expressions of the coördinates are used.

In most theories also the derivatives with respect to all the elements except the mean motion are found without trouble. But the convergence of the series arranged in powers of the mean motions is so slow that even when we have the full numerical values and the algebraical series for a few terms the derivatives with respect to the mean motion are doubtful. Professor Brown's paper gives a method of obtaining these last derivatives accurately. Some further results are also found.

14. Professor Loewy's paper is a continuation of that by the same author published in the January (1903) number of the *Transactions*. A linear homogeneous (finite or infinite) group G with real coefficients is described as real-irreducible when no matrix P of non-vanishing determinant exists such that all the substitutions of the similar group $G' = PGP^{-1}$ have the form

$$\begin{array}{cccccccc}
a_{11} & a_{12} & \cdots & a_{1m} & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & a_{2m} & 0 & 0 & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{m1} & a_{m2} & \cdots & a_{mm} & 0 & 0 & & 0 \\
a_{m+1\,1} & a_{m+1\,2} & \cdots & a_{m+1\,m} & a_{m+1\,m+1} & a_{m+1\,m+2} & \cdots & a_{m+1\,n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{n1} & a_{n2} & & a_{nm} & a_{n\,m+1\,1} & a_{n\,m+1\,2} & & a_{nn}
\end{array}$$

the a's being all real. The following theorem is established: "If a real-irreducible linear homogeneous group is not absolutely irreducible, it is similar to a decomposable group, the matrices of whose substitutions have the form

$$\begin{array}{cccccc}
c_{11} & c_{12} \cdots & c_{1m} & 0 & 0 \cdots & 0 \\
c_{21} & c_{22} \cdots & c_{2m} & 0 & 0 \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c_{m1} & c_{m2} \cdots & c_{mm} & 0 & 0 \cdots & 0 \\
0 & 0 \cdots & 0 & c_{11}^0 & c_{12}^0 \cdots & c_{1m}^0 \\
0 & 0 \cdots & 0 & c_{21}^0 & c_{22}^0 \cdots & c_{2m}^0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 \cdots & 0 & c_{m1}^0 & c_{m2}^0 \ldots & c_{mm}^0
\end{array}$$

the c_{ik} and c_{ik}^0 being conjugate imaginaries. Since a real-irreducible group is under consideration, all the coefficients c_{ik} in the matrices of all the substitutions cannot be real." This theorem, in combination with the results of the previous paper in the *Transactions*, serves as a basis for the study of the real-irreducible constituents of any real linear homogeneous group.

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