(1) involves certainty that the imaging process shall not fail even though as yet perhaps unknown considerations may demand that it be endlessly performable. Accordingly (1) involves a statement included in Poincaré's axiom, which appears indeed to be a presupposition of all logical discourse, the existence of the infinite being unavoidably however unconsciously assumed and so not demonstrable.

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A GERMAN CALCULUS FOR ENGINEERS.

Hauptsätze der Differential- und Integral-Rechnung. Von DR. ROBERT FRICKE. Dritte umgearbeitete Auflage. Braunschweig, Vieweg, 1902. 4vo., 218 pp.

WHILE the needs of American technical schools, and their environment, render a foreign book on the calculus unsuitable for use as a text, the difficult questions which arise in regard to the methods of presentation of this subject are largely the same throughout the world. It is a mistake to imagine that the German brain, for instance, is constructed so differently from the American, that the German *Fuchs* can grasp niceties of the calculus which necessarily escape the American Sophomore. Nor is it logical to presume that the tasks of an engineer differ materially in the two countries. The problems to be fought out are generally speaking about the same, aside from certain minor matters which depend upon traditional systems of instruction. The battle which is being waged on German soil for the closer union and more complete understanding between mathematicians and engineers, is therefore of almost equal interest to the same two classes in America. But we must here pass over the immense amount of fruitful material, which is the product of some of the most eminent minds of Germany*— among them Felix Klein — and which throws strong light on "the necessary and sufficient amount of calculus for the engineer."† It should be remarked, however, that the book which forms the subject of this review is produced, for use in a technical school, in the light of all this inspiring criti-

* See, *e. g.*, the recent files of the *Jahresbericht der Deutschen Mathematiker Vereinigung.*

† Fricke, preface.

cism, and by an author whose fitness for the work possibly exceeds that of any other German mathematician.

Although the present edition is the third, the book will be reviewed as an independent work, both because no previous review has appeared in the BULLETIN, and because the present edition, being the first purposely prepared for use outside the Technische Hochschule in Braunschweig, is considerably revised in form and content.

The author has been considerably influenced, of course, by the special needs of his own school, since the book was originally published for use in that institution alone; and in particular every effort has been made to meet, intelligently, the exigencies of mathematical instruction in a school for engineers. We shall accept the same standpoint in criticizing the book, and shall lay stress on those parts which affect the teaching of mathematics to future engineers.

As is explicitly stated in the preface, the author seeks to maintain a reasonable standard of mathematical accuracy, but he does not desire to sacrifice to this end the simplicity of treatment and the practicability necessary to a student who intends to pursue engineering. For this reason he intentionally uses geometrical and intuitional proof when rigorous arithmetic demonstration is considered too difficult. While the reviewer believes heartily in this principle, the conjugate statement is certainly the one which needs most insistence in our American texts, for while in the past the German has been too difficult in his rigorous presentation, we in America have erred to the side of leniency. To particularize, statements should not be made which are utterly false as they stand, nor should the treatment be such as to render a further study of mathematics impossible to a student who finds himself especially drawn toward it. And so it happens that this book, designed to meet the needs of engineers, and purposely avoiding the difficulties of rigor by occasional geometrical or intuitional proof, still contains many easy methods of proof and presentation which commend themselves to our writers on the calculus, because they are generally more rigorous than those of our books, and because the results obtained are at least accurately true as stated.

The book consists roughly of four parts: (1) the differential calculus, (2) the integral calculus, (3) differential equations, (4) an appendix on functions of a complex variable; of which the first two each consists of two sections. When we notice that all this work, though containing in its first two parts alone more than any of our elementary texts on the calculus, is compressed into 218 pages, an explanation must be made that the book is intended for use in connection with extensive lectures. The author has explained elsewhere that many examples and explanations involving the use of actual mechanical models, as well as geometrical illustrations and proofs, are given in the lectures at Braunschweig to which this book is a companion text. That the last two parts extend the field of the book far past that of our texts is apparent.

The first sixteen pages are devoted to an introduction, in which the notions of "variable," "function," "limit," "continuity," etc., are explained. The notable features of these pages is the accuracy with which these ideas are defined and explained, although the author uses geometrical ideas continually. That reasonable accuracy *can* be combined with simplicity and concreteness is best illustrated by the explanation of a continuous variable by means of the intuitional concept of motion. The word "limit" is explained in a perfectly rigorous way, which is after all simpler than the usual explanations of our texts. The fundamental character of this concept for the calculus appears fully to justify a demand for complete rigor at this point. As an application, the *existence* of the number

$$l = \operatorname*{limit}_{n=\infty}\left(1 + \frac{1}{n}\right)^n$$

is proved in a manner which can be commended even to the advanced student of mathematics as a model of simplicity and accuracy.

But the main portion of this introduction is devoted to the discussion of certain functions which are termed "elementary." These comprise: (a) the rational and irrational ("elementary algebraic") functions, which are defined as those functions which arise from x and constants by the successive application of the four elementary operations and the extraction of roots; (b) logarithmic and exponential functions; (c) trigonometric functions and their inverses. Practically the whole of the book is based upon these "elementary" functions, and the theorems are stated for them alone. This permits a degree of simplicity in proof and of accuracy in statement which is not attainable when no restriction is laid upon the functions treated. Our text-books would do well to follow some such method.

The first section treats the formal rules of differential calculus. The notion of "derivative" is defined *only* for the "elementary" functions, and only at points where these functions are continuous. Perhaps the only point in the book which merits criticism is the treatment of differentials. There are two definitions given, which are unfortunately not compatible. On page 18 the differentials dx and dy are defined by the equations $dx \equiv \Delta x$, $dy = f'(x) \cdot \Delta x$, where

$$y = f(x), \quad \frac{dy}{dx} = f'(x).$$

This definition, if carefully used, is quite justifiable and logical. The other definition, on p. 17, following a usage which is unfortunately common, defines the differentials as certain "infinitesimal" quantities, which are neither zero nor finite; dx and dy being such "infinitesimal" values of Δx and Δy, respectively. As the reviewer has elsewhere remarked,* this second definition is totally illogical in a treatise which intends to use euclidean geometry, especially if intuitional ideas or ordinary theorems on continuity are to be used in the discussions. For the introduction of such "infinitesimal" quantities expressly violates the axiom of Archimedes, which is fundamental in our ordinary conceptions of continuity. The author has, however, explicitly stated on page 18 that this latter definition is not a correct one.

The derivations of the derivatives of ordinary functions are in general the same as usual. The proof of the fundamental limit,

$$\operatorname*{limit}_{\theta=0} \left(\frac{\sin \theta}{\theta} \right) = 1,$$

on page 22 is possibly worth noting, as an extremely easy, yet reasonably accurate proof.

In this first chapter the hyperbolic functions ($\sinh x$, etc.), are introduced, and their derivatives found. As this departure is made, according to the preface, at the earnest suggestion of engineers, the matter may be worth the notice of American writers for students of engineering.

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* "The Doctrine of Infinity," BULLETIN, vol. 9 (2), no. 5 (Feb., 1903), p. 263.

In the next chapter a simple proof of the binomial formula for a positive integral exponent is given, based upon the relation $e^A \cdot e^B = e^{A+B}$. The proof is at least interesting in its curiousness.

The difficulties of "infinitesimal" quantities are encountered quite sharply on pages 33–35. Notice will be taken only of the non-existence (in an archimedean arithmetic) of the number ϵ of page 34, which "approaches zero continuously without becoming identically zero," and "is infinitely small." That the introduction of such quantities is in any way *necessary* to the treatment of engineering topics, or that it simplifies the study of engineering, is certainly a fallacy based solely upon familiarity and tradition.* Had Newton and Leibnitz hit upon the definition of a derivative as a *limit*, our engineers of to-day would never dream of using concepts which destroy all ordinary notions of the continuity of space and motion.

The next section considers some applications of the formal rules previously obtained. In the first chapter, on maxima and minima, it would seem that simplification might have been made; but the general method of presentation, while not differing essentially from the usual treatment, is unusually easy for a student to grasp.

The introduction of the cycloid and the extensive development of its properties in the next chapter, is noteworthy, and might well be imitated in an engineering calculus.

But the chief merit of this section consists in the third chapter, on infinite series. The treatment is reasonably (but not absolutely) rigorous, and certainly superior to the utter lack of treatment which too often obtains. The "sum" of an infinite series is defined with reasonable accuracy, the "convergence" and "absolute convergence" of series are discussed, and the condition $u_{n+1}/u_n \leqq r < 1$ is established for the convergence of a series of positive terms. Even power series are discussed briefly, but without many proofs of theorems.

The average value theorem is then derived; and is used for a very simple treatment of Taylor's development, which can be recommended as simple, accurate and above all *true*. Even more remarkable, from the standpoint of comparison with the American calculus for engineers, is the investigation of the interval of convergence of the Taylor's expansions, for each of

* That this does not preclude a *proper* definition of *differentials* has already been intimated.

the "elementary" functions considered. It is clearly demonstrated that the method of such investigations, and the proof of the necessity of investigation, is within the reach of a fairly intelligent student. Last of all these comes the binomial theorem, of which very clear proof and statement are given.

The discussion of indeterminate coefficients at the end of the chapter establishes the uniqueness of McLaurin's expansion. It is remarkable to notice that the proof of the uniqueness of the expansion is just what is usually given in our texts, in place of a proof of the formula itself!

The last chapter of the section deals with some common indeterminate forms in a very acceptable manner, though the proofs are not supposed to be entirely rigorous. The use of power series as an alternate process is somewhat out of the ordinary; and the investigation of the relative intensity of the approach of x^n and log x toward infinity, is quite so, in such a text.

The second part, on integral calculus, opens with a treatment of definite and indefinite integrals which is worthy of notice and possible imitation. The accuracy of statement which characterizes the book is noticeable on page 80. The theorem that two integrals of the same integrand can differ only by an additive constant, is referred for proof to the average value theorem of differential calculus. That the author contents himself with the mere reference, and does not give the proof which he shows is necessary, is typical of the attempt at simplicity of presentation. A proof just here would probably confuse the ordinary student.

The definition of a definite integral is given as a sum (which is meant, of course, for the limit of a sum) and the connection with indefinite integrals is established by means of the average value theorem in connection with an admirable geometrical representation, which renders the reasoning concrete and therefore easy. The average value theorem for integrals is then proved and its concrete applications are not wanting in the rest of the book.

Aside from certain ordinary formulæ, the chapter further contains the interpretation of hyperbolic and trigonometric functions by means of areas, and certain formulæ for the approximate calculation of definite integrals. The first topic is of doubtful importance to engineers; but the importance of the second to them can scarcely be overestimated. Such a treatment should be given, even in more detail, in all calculus texts for engineers.

In order to establish the expansion of a rational function into partial fractions, a few theorems on algebraic equations are stated in the second chapter. This enables the author to give a proof of the expansion in question which amounts to something more than a bald assertion. The Lagrange formulæ of interpolation for constructing a function of lowest possible degree taking on a given set of values for given arguments is surely of use to the engineer, and should be found, apparently, in all engineering texts on the calculus.

The remainder of the chapter establishes for the most part usual theorems. It might be remarked that the author's anxiety to introduce elliptic integrals has led him to a statement (page 111) which, without explanation, might lead a student to suppose that such an integral as

$$\int \frac{(2x-1)^2 dx}{(17-3x+6x^2-4x^3)^{\frac{1}{2}}}$$

did *not* represent an "elementary" function.

The fourth section is not especially noteworthy. The usual theorems regarding functions of several independent variables are discussed, and the treatment is not very unusual. There are, however, several minor points where the presentation is especially good, and a little more surface theory than is usual in our books is given. The proof on page 120 of the theorem

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y},$$

where $f(x, y) = 0$ is given, is a clear instance of the necessity of considerable care in the handling of differentials, even when properly defined. Since

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is the definition of dz (which is *otherwise* unknown) it follows that we cannot set $dz = 0$ without an assumption which practically amounts to the theorem itself! The theorem at the bottom of page 121: "Given $y = f(x_1, x_2, \cdots, x_n)$, $x_i = \phi_i(x)$, then

$$\frac{dy}{dx} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx}\text{''},$$

is really the fundamental theorem upon which all the work for functions of several variables might be based.

The third part (Section V) treats of differential equations. A single ordinary equation in one dependent variable, a set of two ordinary equations in two dependent variables, and a single partial equation in one dependent and two independent variables, are the topics treated. The selection of methods of solution is good, scarcely any well known elementary method being omitted. The best features of the section are a geometrical proof of the existence of solutions of an ordinary equation, and geometrical interpretations of the meanings of the several types of equations discussed.

The introduction of the hypergeometric series and its differential equation, at the close of the section, will sufficiently illustrate the scope of the work. Of course the reasoning is necessarily somewhat crude at points in this section, as for instance the statements that a given differential equation can always be written in normal form ; but the work is generally more rigorous, and much more complete, than corresponding discussions in any elementary calculus in the English language. The 38 pages of this section compare favorably with such books as Johnson or Forsyth in giving a general oversight of the subject. In general, the short courses on differential equations given in American technical schools — when such courses are given at all — lay too much stress on ordinary equations to the total or partial exclusion of partial equations. For the problems of higher engineering very often lead to partial differential equations, and the solution of many other problems usually solved by cumbrous methods can often be effected in simple form by the use of such equations.

The short appendix on functions of a complex variable is scarcely to be mentioned for imitation in our books on calculus. Important as is the subject matter for the engineer and for the mathematical student, such work is beyond the average teacher in our technical schools, and could not be given to advantage.

For the purposes of our schools it would seem that more time should be spent on applications of the calculus to mechanics. These applications are quite as immediate as the applications to geometry, for a derivative is represented by a velocity quite as truly and simply as by the slope of a curve; and the second derivative finds easier direct representation in acceleration than in curvature. Moreover the subject of mechanics is of more importance to the engineer than the *extended* study of curves and surfaces. Of course the geometrical interpretation of the calculus must continue to be given quite as fully, at least, as is now common; but it would certainly be useful to introduce the ideas of mechanics early in the work, and bridge the chasm between formal calculus and engineering mechanics by as many applications of the calculus to mechanics as possible.

The remarks which have been made have to do, for the most part, with the method of presentation of the calculus to future engineers in American technical schools. This is a most difficult theme, and it is scarcely hoped that the opinions which have been expressed will be generally accepted as final. Many important topics have not been discussed, since they were not suggested by the book in hand, but it is hoped that the discussions of the matters which have been touched upon will prove suggestive and possibly beneficial in the arrangement of the work in calculus in our technical institutions, and in the preparation of texts designed for such schools. The whole subject surely merits considerable careful thought and thorough discussion, particularly in view of the exceptional growth of institutions especially intended for students of engineering, and in view of present lack of any text-book on the calculus expressly designed for students of engineering which really meets their needs.

Herr Fricke's book cannot be said to be the ultimate ideal of such a text, and there are many things which would need entire revision to make the book suitable for American schools. But a great advance is marked in it over our American texts, in the combination of simplicity and concreteness with reasonable accuracy; and on this account it may well serve to suggest to the instructor or to the author of a text on the calculus, possibilities of presentation of the subject to students of engineering.

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YALE UNIVERSITY,
December, 1902.