index, and the moderate price, which all yield so much to the usefulness of the present edition, may also be a feature of the French edition. EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE,
PARIS, *December*, 1902.

FORSYTH'S DIFFERENTIAL EQUATIONS.

Theory of Differential Equations. By A. R. FORSYTH. Part II. *Ordinary Equations, not Linear*, Volumes 2 and 3; Part III. *Ordinary Linear Equations*, Volume 4.

IT becomes necessary from time to time to sum up in a work of considerable volume the knowledge which has accumulated in a certain field. The theory of differential equations is in some respects the most important part of mathematics. It is in this field that the astronomer and physicist most frequently appeals to the mathematician for assistance; for his problems, when finally formulated, usually assume the form that a certain differential equation is to be integrated. The most important transcendental functions, too, have been furnished to the mathematician by the integration of differential equations. No wonder then, that the literature is extensive, and there can be no doubt that mathematicians will feel grateful to Professor Forsyth for having lightened for them the labor of becoming acquainted with the labyrinth of investigations which have been carried on in this field.

That Professor Forsyth should have chosen to treat the linear equations last, may have been due to the fact that other works existed which treat of them in a modern and adequate manner. But systematically, and historically they should come first, as almost every question in regard to non-linear equations, that has been answered, has been suggested by the theory of linear equations. A student, therefore, would do well to read volume 4 of the present work first. Fortunately volume 4 has been so written as to enable him to do so. But we cannot pass this distinction between linear and non-linear equations by without remarking that there is no better way to convince ourselves of our ignorance on the subject of non-linear differential equations, than by studying Part II of Professor Forsyth's book. We find here, gathered with the greatest erudition, practically all that is known about the subject. All of this covers a great many pages, the analysis is terrific, and the insight gained is very small indeed. One feels that new principles are needed, the old ones work so slowly. Some day some one will have an inspiration, and some order will be brought into this chaos.

But in the meantime we must get along as well as we may and Professor Forsyth's book serves well as a guide over this intricate territory. It opens with the consideration of the various forms to which a system of differential equations may be reduced, and then proves in Chapter II Cauchy's fundamental theorem on the regular integrals of such a system in the vicinity of an ordinary point. The distinction between regular and other integrals is well emphasized. But at least one of the proofs of the fact that the regular solutions are uniquely determined by the initial conditions, seems trivial, as Cauchy's proof of the existence of the regular solutions is also sufficient to establish their uniqueness.

In Chapter III the author takes up and classifies the various kinds of non-ordinary points of a differential equation of the first order and of the first degree in the derivative,

$$\frac{dw}{dz} = f(w, z).$$

When the function $f(w, z)$ is uniform, Weierstrass's theory of uniform functions of several variables enable one to do this satisfactorily. Other cases are wisely relegated to a later chapter.

If $w = a$, $z = c$ is such a non-ordinary point of the function $f(w, z)$, the question naturally arises as to the form of the solutions of the equation in the vicinity of such a point. This question is therefore discussed in the next chapters, leading of course, to the classical theorems of Briot and Bouquet. Chapter IV discusses this question for accidental singularities of the first kind, introduces the notion of a point of indeterminateness due to Fuchs, and closes with Fuchs's condition for the uniqueness of Cauchy's regular integral. For, while this integral is clearly the only *regular* integral determined by the initial conditions, there may be in general other integrals, not regular, satisfying the same initial conditions.

In Chapter V the differential equation is reduced to typical forms valid in the vicinity of an accidental singularity of the second kind, and in Chapter VI the form of the integrals in the vicinity of such points is examined.

In Chapter VII the essential singularities of the differential equation are taken up. The most important result of this chapter is Painlevé's theorem : that the points of indeterminateness, in particular the essential singularities of the integral of the equation

$$\frac{dw}{dz} = f(w,z),$$

where f is rational in w and uniform in z, are fixed points determined by the equation itself. The author takes care to emphasize that there is here an essential distinction between equations of a higher, and those of the first order.

If the differential equation is algebraic, of degree n, in dw/dz, it becomes necessary to consider branch points. This is done in Chapter VIII. Painlevé's theorem is extended to cover this case. In Chapter IX some very important investigations are reproduced, due to Fuchs and Poincaré. The question proposed by Fuchs is this : what differential equations of the first order are those, the branch points of whose integrals are fixed, *i. e.*, independent of the constant of integration ? Fuchs gave the conditions under which this takes place in a very elegant theorem. Poincaré then showed that all such equations can either be transformed into a Riccati's equation, or else are integrable by quadratures or algebraic functions, so that such equations do not, as was at first expected, lead to new transcendental functions.

Chapter X contains familiar matter. It treats of the equations of the first order whose integrals are uniform, and also gives a few theorems on equations with algebraic integrals.

In the third volume the same investigations are to a certain extent carried out for systems of differential equations and in particular for equations of the second order. Chapter XVII is especially interesting, as it gives the proof of Bruns's theorem that the only algebraic integrals of the problem of n bodies are the well-known ones regulating the uniform motion of the center of mass along a straight line, the law of energy, and the laws of area. Professor Forsyth has simplified the proof somewhat and completed it in some details.

Volume IV is devoted to linear differential equations. The first chapter furnishes the existence theorem as proved by Fuchs, and of course, as a corollary, the fundamental theorem that all of the singularities of the solutions are fixed. There follows in Chapter II the discussion of the properties of a fundamental system in the vicinity of a singular point. This leads to the consideration of the fundamental equation and its invariance. A canonical fundamental system is set up, made up of solutions arranged in groups. These again are arranged in subgroups, according to Hamburger. The whole theory is based, as it must be, if it is to be complete, upon Weierstrass's theory of elementary divisors. This latter theory is briefly presented, but in a form which is none too lucid. The fact that the constant term of the fundamental equation becomes equal to unity, when the equation has been so transformed as to be devoid of the term involving the next to the highest derivative, is referred to on page 40 and throughout the book as Ponicaré's theorem. To the present reviewer this practice is distasteful. Poincaré is certainly one of the foremost among living mathematicians, and his name is attached to discoveries of the most fundamental importance in all domains of analysis. But this theorem (so-called) is of the most trivial character. Nobody could possibly help noticing it, and Poincaré's name should be reserved for some more substantial theorem.

In Chapter III we find the conditions that an equation shall have all of its integrals regular at a given singularity, $x = a$, viz., that the equation shall have the form

$$\frac{d^m w}{dz^m} = \frac{P_1}{z-a}\frac{d^{m-1}w}{dz^{m-1}} + \frac{P_2}{(z-a)^2}\frac{d^{m-2}w}{dz^{m-2}} + \cdots + \frac{P_m}{(z-a)^m} w = 0,$$

where $P_1, \cdots, P_m$ are holomorphic functions of $z - a$ in the vicinity of $z = a$. The converse of this theorem is then established by the method of Frobenius, which furnishes at the same time a method for constructing all such regular solutions. Professor Forsyth makes some remarks about the necessity of proving the converse, which seem to the reviewer to be quite unnecessary and even misleading. That the converse of a theorem requires proof, is something with which most of us are acquainted, and the impression is created as though Fuchs did not know this, while he actually did prove the converse in his first paper on the subject, reference to which is made later. The rest of Chapter III is taken up with the discussion of the conditions under which all of the solutions in the vicinity of $z = a$ are free from logarithms. This is well presented.

In Chapter IV the equations of the Fuchsian type are discussed, *i. e.*, those which have all of their solutions regular in the vicinity of every singularity, including infinity. In a note on page 128 the author makes a peculiar slip. The general form of an equation of the Fuchsian type is

$$w^{(m)} = \frac{G_{\rho-1}}{\psi} w^{(m-1)} + \frac{G_{2(\rho-1)}}{\psi^2} w^{(m-2)} + \cdots + \frac{G_{m(\rho-1)}}{\psi^m} w,$$

with

$$\psi = \prod_{k=1}^{\rho} (z - a_k),$$

where $a_1, a_2, \cdots a_\rho, \infty$ are the singular points and G_λ denotes a polynomial of degree not higher than λ, and where λ must be a positive integer or zero. The author puts $\rho = 0$ and concludes that the equation then has constant coefficients. This is obviously false. Moreover an equation with constant coefficients actually does not belong to the Fuchsian class.

Having noted that a linear differential equation of the second order of the Fuchsian type with three singular points is completely determined by the assignment of these points and the exponents belonging to them, the author proceeds to discuss them. This is, of course, the theory of Riemann's P function, which is historically the source of the whole general theory, a fact which the author does not seem to mention. In connection with this, the theory of Gauss's equation naturally appears and the twenty-four solutions due to Kummer are easily deduced. Other equations of the second order and of the Fuchsian type are then discussed, largely after Klein. On page 153 the author omits to state that $\rho_1 + \rho_2 + \cdots + \rho_n$ must be an integer if $z = \infty$ is not to be a singularity of w. He then takes up the equation with five singular points and proves Bôcher's theorem that when the five points are made to coalesce in all possible ways, each limiting form of the equation is equivalent to one of the differential equations of mathematical physics. Finally in this chapter the question of polynomial and rational integrals is discussed.

Chapter V is devoted to equations of the second, third and fourth orders with algebraic integrals. For the equations of the second order this leads, of course, to the polyhedral functions. One cannot help feeling that the treatment of this problem for the equations of the third and fourth orders is in-adequate. The author makes no use of the Picard-Vessiot theory of linear differential equations, which is of primary importance in such discussions, nor does he make much use of the theory of invariants and covariants which is also of fundamental importance in the consideration of these and other questions. This is, of course, due to the author's intention to lay stress primarily upon the function-theoretic phases of the theory of differential equations. He himself has made important contributions to the theory of invariants and covariants of linear differential equations, and it appears to the reviewer a case of too much self-denial on the part of the author not to have given more of it in this book. In fact, a great deal remains to be done in this field, and it would have been valuable to have pointed this out to future investigators.

The case where all the solutions of an equation are regular near a singular point having been treated, the next chapters are naturally devoted to the methods for obtaining such solutions as are regular near a singular point when not all of them are, and the far more difficult problem of determining the non-regular solutions. Here there is still a large field for investigation.

An integral of a differential equation may be of the form

$$e^{\Omega}x^{\rho}\psi(x)$$

in the vicinity of $x=0$, where ρ is a constant, $\psi(x)$ a holomorphic function of x, and

$$\Omega = \frac{a_1}{x^s} + \frac{a_2}{x^{s-1}} + \cdots + \frac{a_s}{x},$$

so that it differs from a regular integral only by the presence of the factor e^{Ω}. Such an integral, if it exists, is called a normal integral. There may also be integrals of a similar form in which however $x^{1/k}$ appears in place of x where k is a positive integer. They are called subnormal. The conditions for the existence of normal and subnormal integrals are investigated, but none of these investigations are as yet in a final form. Moreover these normal and subnormal integrals are clearly nothing but very special examples of non-regular integrals. A general theory on such a basis requires a systematic and complete theory of essential singular points of uniform functions of a complex variable. Such a theory however does not exist.

It is necessary therefore to change the method of attack. The general theory shows that in the vicinity of the singular point $x = 0$, a solution exists of the form

$$x^\rho \psi(x)$$

where $\psi(x)$ is in general a Laurent series. The question is this: how to determine the exponent ρ and the coefficients of $\psi(x)$. In the regular case, when $\psi(x)$ is an ordinary power series, substitution of this expression into the differential equation, and comparison of powers of x, solves the problem. One may do the same thing in general. But then one finds it necessary to solve a system of linear equations infinite in number and with an infinity of unknown quantities. This leads to the notion of infinite determinants, due primarily to G. W. Hill. It is another example of a fact noticeable in the history of mathematics that the fearless application of general principles, regardless of rigor, frequently leads to new and important notions. Hill applied infinite determinants just as though they were finite, paying no attention to convergence or rigorous definitions. Of course that is not mathematics. But after all, the notion once placed before the world, it was not difficult to make it precise by the application of the theory of limits and rigorous reasoning. This was done by Poincaré, and the method of Hill was then applied to the general case by von Koch. Chapter VIII is devoted to this theory.

In Chapter IX the theory of linear differential equations with uniform simply or doubly periodic coefficients is developed. A large part of this theory might have been written down as a mere corollary of the general theory by means of the substitution

$$t = e^{2\pi i z/w}$$

which the author mentions later (p. 422) in connection with special cases. There is a special section devoted to Lamé's equation.

In the last chapter we find some account of equations with algebraic coefficients, and a brief sketch of the theory of automorphic functions. The theorem that it is always possible to find an auxiliary variable, such that the dependent and independent variables of a linear differential equation can be expressed as uniform functions of it, for instance as Fuchsian and Zetafuchsian functions, is one of the most important in the theory of functions. The general principle upon which this theorem rests should be permanently associated with the name of Poincaré.

In the execution of a task of this magnitude, it is inevitable that some errors should creep in. The reviewer is astonished at their small number. The salient features of almost every branch of the subject are presented, on the whole in a lucid manner. Some not inconsiderable improvements have been made by the author himself. He has undertaken an arduous task, and has accomplished it well. For the assistance which he has thereby given to a general understanding of this great department of mathematical learning he is entitled to our gratitude.

E. J. WILCZYNSKI.

NOTES.

A NEW edition of the Annual Register of the AMERICAN MATHEMATICAL SOCIETY will be issued in January next. Blanks for furnishing necessary information have been sent to each member. A prompt response will greatly facilitate the work of the Secretary.

THE October number (volume 25, number 4) of the *American Journal of Mathematics* contains: "The plane geometry of the point in point space of four dimensions," by C. J. KEYSER; "On the functions representing distances, and analogous functions," by H. F. BLICHFELDT; "Surfaces whose lines of curvature in one system are represented on the sphere by great circles," by L. P. EISENHART; "On the invariants of a homogeneous quadratic differential equation of the second order," by D. R. CURTISS; "Surfaces of constant mean curvature," by L. P. EISENHART.

ACCORDING to the annual list published in *Science*, 266 doctorates were conferred by American universities during the academic year 1902–1903. Of the 136 distributed among the "sciences," the following 7 are recorded with theses in mathematics: H. A. CONVERSE, Johns Hopkins University, "On a