

cal excess of the inscribed polygon. By subdividing the region by meridians and parallels, and allowing the subdivisions to approach zero, the solid angle is shown to equal the spherical excess.

If R is a closed surface,

$$\Theta(O, R) = 4n\pi,$$

where n is a positive or negative integer, or zero. *The number n is defined as the ORDER of the point O with respect to the surface.* The order of O with respect to a closed surface is equal to the order of O with respect to a plane section through O , when this is defined. The remainder of the proof that a simple regular closed surface divides space into two continua, of each of which the surface is the total boundary, is similar to that in two dimensions. The method here used is equally applicable to space of n dimensions.

UNIVERSITY OF MISSOURI,
October, 1903.

THE THEORY OF WAVES.

Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique. Par JACQUES HADAMARD. Paris, A. Hermann et Fils, 1903. xiii + 375 pp.

IN handling mathematics and especially mathematical physics, in which the data are never quite so extramundane as in some branches of pure mathematics, there are two things which serve to determine the correctness of results. One is intuition; the other, rigorous accuracy in analysis. The great investigator like Gibbs, even when dealing with such critical questions as arise in statistical mechanics, never has need of epsilons and deltas to insure the results against error. There are others of us however to whom the rigorous proof is more convincing and even necessary to conviction. Furthermore, as one proceeds from the more evident to the more refined phenomena of physics, the need of exact demonstrations becomes constantly greater. M. Hadamard has shown us in the *Transactions* * an example of what may be obtained from the critical rigorous

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treatment of the theory of vibrating elastic plates. In his most recent work, *Leçons sur la propagation des ondes*, he treats the more general problem of waves in any medium and of sound waves in particular. A first notice and résumé of the work was inserted in the *Bulletin de la Société mathématique de France* in 1901 under the title "Sur la propagation des ondes." Later Appell in the third volume of his *Traité de mécanique rationnelle*, pages 296–318, treated the elements of the theory, and to this presentation we may refer those who would have a first glimpse into the kinematic part of the subject.

When one thinks of it, the wonder is that the processes of strict mathematical rigor are at all applicable to physics. The atomic structure of matter renders the exact application of epsilon proofs impossible. That which takes place in reality in the evaluation of most physical quantities is not unlike the behavior of those divergent series which appear to converge for a certain number of terms and which up to that point give closer and closer approximations to the value desired but afterwards diverge rapidly from it. Yet so fine is the structure of matter that the errors made in applying those methods which would be strictly applicable to a continuous distribution are not many nor large. It is only in such delicate effects as those connected with the dispersion of light that the discontinuities in structure become evident.

In 1860 Riemann published a memoir entitled "Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite," in which he signalized the existence of waves of discontinuity. By a wave of discontinuity is meant a surface moving through the medium and such that the coördinates of the points of the medium or their derivatives of some orders experience abrupt changes in value when passing from one side of the surface to the other. In 1877 Christoffel took up this work in the *Annali di Matematica* and generalized it to three dimensions, limiting himself on account of analytic difficulties to the case of waves of infinitesimal discontinuity of the velocity (waves of percussion, *ondes de choc*). Later, 1885–7, Hugoniot without the knowledge of the work of either of these mathematicians investigated the subject, found most of the results of his predecessors, rendered more precise the idea of *compatibility* in the case of plane waves, and introduced numerous improvements. Now Hadamard in the treatise before us extends these results to their practical completion. Incidentally

he has given us a large number of accessories either entirely new or worked out better than in previous presentations. The work covers the ground of his lectures at the Collège de France during the years 1898–1900. Although the present imperfect state of our knowledge forces the author to leave certain points for future investigations, we may rest assured that the work as it stands is quite up to date including, in addition to the author's personal contributions, much that has been published in the years which have intervened between the date of delivery of the lectures and their publication.

In the subject of harmonic functions, Green's functions, and their determination from given boundary conditions, three distinct problems must be recognized. First is the problem of Dirichlet to determine a harmonic function V when given the value of V at all points of the boundary (and its behavior at infinity if the domain of the function is exterior to the bounding curve or surface upon which the values of V are given). Riemann indicated the method of establishing the existence of the solution and numerous subsequent authors have filled in the details. The second problem rather than the first concerns the motion of fluids. In this case are given not the values of the function V but those of its normal derivative dV/dn over the boundary in question. For the existence of the solution of this so called problem of Neumann, Lord Kelvin has indicated a proof analogous to that given by Riemann for the former problem. Numerous objections, however, may be advanced. The third problem, which Hadamard calls the *mixed* problem, is that which occurs most frequently in the theory of fluids whether real or fictitious like heat. Here the values of $dV/dn - k^2v$, where k varies from point to point on the boundary, are given. In particular the value of V may be given for part of the boundary and the value of the derivative dV/dn for the rest.

In the first chapter of 57 pages the author gives an excellent detailed exposition of the problem of Neumann, the solution of which is less perfected and less widely known than that of Dirichlet's problem. The case of the plane and that of space are separately taken up. A modification of the function of Neumann is introduced and applied to effecting the solution of the problem in the cases of one sphere and of two concentric spheres. It is shown at the close that no considerations analogous to those by which the problem of the spheres has just

been solved will apply to other surfaces and hence other methods must be invented. As to the mixed problem, very few results are known. With the statement of these the subject is left.

The second chapter, which is 71 pages in length, deals with waves from the standpoint of kinematics. Here Hadamard introduces three material improvements over the work of Hugoniot: first, by carefully separating from the kinematic part the dynamic part which depends on the physical properties of the medium; second, in rendering precise the idea of compatibility for motions whether plane or not; third, by the remarkably simple geometric interpretations which he gives to the analysis. The greater part of the work is his own investigation. The importance of the subject and the fact that Hadamard's predecessors have been so completely disregarded by mathematicians the world over demand of this review a rather long summary of the ideas which lie at the basis of the present work.

Let a, b, c be the coördinates of the points in the medium chosen with reference to some initial position and remaining unalterably attached to their respective points during all the motion. Let x, y, z be the coördinates in space of these same points at any instant t . We have

$$x, y, z = F_1, F_2, F_3 (a, b, c; t).$$

The impenetrability of matter affords the condition that, given a set of values for x, y, z , the three equations must admit of a singly determinate solution for the coördinates a, b, c . The further supposition is made that x, y, z are in general continuous and differentiable with respect to the four variables $a, b, c; t$. From the purely physical side this latter assumption is by no means obviously justifiable; but it is found to give a sufficiently good account of the phenomena. It may be noticed that to introduce a discontinuity with respect to t itself corresponds to an abrupt change in position and a consequent infinite ratio between the force and the mass. The same is true if there be a discontinuity of finite magnitude in the derivatives with respect to t . The explanation of the fact that a surface of discontinuity of the velocity may exist without infinite forces is that the mass in the surface at any instant is vanishingly small. To introduce a discontinuity with respect to a, b, c and the corresponding derivatives is exactly what the physical properties of

the fluid are constantly doing by the process of diffusion. But as the rate of diffusion is very small with respect to the rate of propagation of the wave of discontinuity the conditions imposed by Hadamard upon the functions F_1, F_2, F_3 may be accepted. For example, as a corollary of these assumptions if a portion of the fluid be once in contact with the wall of the containing vessel it must remain always in contact with it. From the point of view of the kinetic theory of gases this is the grossest sort of untruth when we fix our attention upon any individual molecule. If, however, we consider the molecules near the wall in their totality we see that several seconds would be required for them to diffuse away by so much as a few centimeters into the surrounding gas, whereas the waves of discontinuity are propagated off with a speed 10,000 times as great.

The author regards the stretched shear as a fundamental displacement. In the shear proper all points are displaced in the same direction, parallel to the fundamental or fixed plane, and by an amount proportional to their distance from that plane. In the simple stretch the points move normally to the fixed plane by an amount proportional to their distance from the plane. In the stretched shear all points move parallel to a certain vector by an amount proportional to their distance from a given fundamental plane. If a surface instead of a plane be fixed identically and if the displacement be otherwise arbitrary, the same statement may be made concerning the first order part of the infinitesimal displacements in the immediate neighborhood of any point of the surface. If $f(a, b, c) = 0$ be a surface which remains fixed point for point, and if at the point a^0, b^0, c^0 the first non-vanishing derivatives of x, y, z with respect to a, b, c are of the n th order, then the deformation in the neighborhood of a^0, b^0, c^0 is said to be of the n th order. Hadamard shows that in this case also the principal part (which is of the n th order) of the displacement may be represented by a vector \mathbf{l} , parallel to which the displacement takes place and by an amount $D^n/n! \mathbf{l}$ proportional to the n th power of the distance from the surface.

Omitting the short treatment of the classic results concerning velocities and vortices in fluids, we come to the subject of discontinuities. Suppose that Φ is a function of $x, y, z; a, b, c; t$ and of the derivatives of all orders of x, y, z with respect to $a, b, c; t$. Furthermore suppose Φ and its derivatives with

respect to a, b, c ; t are in general existent and continuous. Let S be a surface of discontinuity such that the function in question and its derivatives approach definite values when the point for which they are evaluated approaches the surface S from either side, but suppose those values to be different upon opposite sides of S . It is shown that if Φ_1 and Φ_2 be the values of the function on the two sides of the surface, then the formula for the total differentials of Φ_1 and Φ_2 along the surface apply as if there were no discontinuity. Further if the derivatives of the n th order of Φ with respect to a, b, c are the first which exhibit the discontinuity, the changes which these derivatives experience are proportional to the values of the corresponding n th order derivatives of the function $f(a, b, c) = 0$ which represents the surface. To apply this to the study of the motion of any medium the function Φ is taken to be any one of the derivatives of x, y, z with respect to $a, b, c; t$. The order of the discontinuity is the order of the derivative of lowest order exhibiting a discontinuity. The index of any derivative is the order of the derivation with respect to t in that derivative. Thus in the consideration of a discontinuity of the n th order derivatives of index $0, 1, 2, \dots, n$ occur. The author shows that completely to determine a discontinuity of the n th order there are required $n + 1$ vectors $\mathbf{l}, \mathbf{l}_1, \dots, \mathbf{l}_n$ of which any one \mathbf{l}_i serves to determine the changes in the derivatives of index i .

At this point the fundamental notion of *compatibility* is introduced. Let the equation of the surface of discontinuity referred to the medium itself be $f(a, b, c; t) = 0$. In case the variable t does not occur, the discontinuity will always affect the same molecules and is called *stationary* even though it may and in general will change its position in space. That the two motions on the two sides of a stationary discontinuity be compatible it is evidently necessary that the normal components of the velocity, the acceleration, and all the accelerations of higher order perpendicular to the surface $f = 0$ be the same upon both sides of the surface. Otherwise either the law of impenetrability will be violated or the medium will separate along the surface, leaving cavities between the two portions of the medium. In case t does occur in the expression for f , the surface will at successive instants of time affect different molecules. Such a surface will be called a *wave of discontinuity* or simply a *wave*. For waves the condition of compatibility is

that the wave surface rest unique. This amounts to establishing a relation between the vectors $\mathbf{l}, \mathbf{l}_1, \dots, \mathbf{l}_n$ and the velocity of propagation θ . The author finds that the necessary and sufficient kinematic conditions of compatibility are given by the equations

$$\mathbf{l} = \frac{\mathbf{l}_1}{-\theta} = \frac{\mathbf{l}_2}{(-\theta)^2} = \dots = \frac{\mathbf{l}_n}{(-\theta)^n}.$$

In addition to these there are in any given medium dynamical conditions of compatibility to be imposed upon the vector \mathbf{l} and the rate of propagation θ . The velocity of the propagation of the wave in space is $\theta + v_n$ where v_n is the component of the velocity of the molecules perpendicular to the wave surface.

From the kinematic point of view we have associated with a wave the vector \mathbf{l} , the unit normal \mathbf{n} perpendicular to the wave front, and the velocity of propagation θ . The ratio of two successive densities or the first non-vanishing derivative of that ratio contains as a factor the scalar product $\mathbf{l} \cdot \mathbf{n}$ of the vectors \mathbf{l} and \mathbf{n} . The variation of the curl or the double of the molecular rotation is $\theta \mathbf{l} \times \mathbf{n}$, the product of the velocity of propagation into the vector product of \mathbf{l} into \mathbf{n} . The wave of discontinuity is said to be longitudinal if \mathbf{l} and \mathbf{n} are parallel; transversal if \mathbf{l} and \mathbf{n} are perpendicular, that is if the discontinuity vector lies in the wave front. From this it follows that a longitudinal wave does not affect the curl; nor a transversal wave, the density (divergence). A short treatment of the effect produced by a discontinuity of order n upon the derivatives of order higher than n closes this remarkable chapter.

After this elaborate discussion of the kinematics of the problem, follows a short chapter of 14 pages containing the formulation of the dynamic problem in case of fluids whether liquids or gases. Passing over the establishing of the equations of motion and the discussion of the physical equation which connects the pressure, temperature, and density, we come to the essential idea underlying the whole work. The motion of a fluid is determined by three sets of data: first the equations of motion, second the initial conditions which state the positions and velocities of the particles at the initial instant $t = t^0$, third the boundary conditions which include the motion of the walls of the containing vessel and the pressure p at all points of the free surfaces. In the case of liquids these conditions allow us to calculate the value of dp/dn at every point of the containing

vessel. If there are no free surfaces the problem of the solution becomes the problem of Neumann which was treated at length in the first chapter. If there are free surfaces we have the "mixed" problem concerning which little more is known than that it has a unique solution provided the solution exists. The results arising from discontinuities in the given conditions are taken up in the fifth chapter. In the case of gases matters are far different. The given conditions are redundant or contradictory. For on the assumption that the temperature is constant or known, the relation connecting the pressure and the density may be solved so as to give the pressure in terms of the density. The initial positions of the molecules determine the density, and hence the pressure at each point of the gas including of course the points in contact with the walls. The accelerations may therefore be computed from the dynamical equations. On the other hand the accelerations of the points in contact with the walls are among the arbitrary given conditions. These accelerations are in general different and hence a contradiction arises. For example, in the case of small rectilinear motions in a gas we obtain with Riemann the equation

$$\frac{d^2x}{dt^2} = \theta^2 \frac{dx^2}{d\alpha^2}.$$

The first member of the equation states the value of the arbitrary acceleration which we choose to impart at the initial time to the wall of the vessel. The second member depends solely upon the interior state of the gas at that initial instant. These cannot usually be equal. The apparent contradiction is explainable by the fact that at the initial instant a wave of discontinuity of the second order leaves the wall and is propagated into the interior of the gas.

This whole subject of the rectilinear movement of a gas is discussed to the length of 82 pages in the fourth chapter. The author reproduces with important changes, comments and additions the work of Riemann and Hugoniot. His unusual analytic proficiency enables him to connect in a wonderful manner the physical problem of the propagation of waves and the mathematical problem of Cauchy concerning the characteristics of partial differential equations of the second order. We pass by the work on differential equations and give in résumé some of the interesting properties of the waves. Suppose the gas con-

fined to a cylindrical tube into which is fitted a movable piston. In general at no instant will there be compatibility between the interior conditions and the motion of the piston, and hence at each instant a wave of the second order, that is a wave of discontinuity of the acceleration, will be sent off. This wave is propagated with the velocity of sound. Now if the piston is driven along with an acceleration the consecutive waves will overtake each other and will produce a phenomenon much like that we see when a boat is driven rapidly through the water. The little waves running out from the bow overtake each other, pile up and break at the distance of a few feet from the bow. To return to the case of air, let us suppose that for a certain interval of time prior to the instant t_1 the acceleration of the piston has been so chosen that the waves of the second order shall come together at the time T . At the instant t_1 let the acceleration of the piston be diminished so that from that time on the waves will cease to overtake each other at the time T . There will then arise at the instant T a discontinuity of the *first order*, that is, an abrupt change in the velocity of the molecules. The discussion of these waves of the first order (waves of percussion or *ondes de choc*) reveals the fact that they have a totally different rate of propagation from the waves of the second order — the rate may be indefinitely great under proper circumstances. The results derived theoretically are admirably confirmed by the experiments of Vieille (1898–1900). By the use of explosives so great an acceleration may be communicated to the air that the waves of the second order propagated at the velocity of 330 meters per second overtake each other in the distance of a few centimeters. The wave of the first order which is thus formed is propagated with a velocity of over 1,200 meters per second, or about four times the velocity of sound. In this connection it is interesting to speculate as to whether in such violent disturbance as thunder that first distinctive *click* which is sometimes heard before the general body of sound may not be due to a discontinuity in velocity. The following rumble seems to have the characteristics of a highly heterogeneous combination of waves.

At this point is raised the objection of Hugoniot. The law of gases which has been assumed is the adiabatic law of Poisson

$$p = \kappa \rho^m \quad \text{or} \quad \frac{p_1}{p_2} = \frac{\rho_1^m}{\rho_2^m}$$

(where m is the ratio of the specific heats — about 1.40 in air). This law is deduced on the assumption that the gas is practically at rest. From certain physical considerations Hugoniot is led to substitute the relation

$$\frac{p_1}{p_2} = \frac{(m+1)\frac{\rho_1}{\rho_2} - (m-1)}{(m+1) - (m-1)\frac{\rho_1}{\rho_2}}$$

for gases subject to violent motions. The difference between these two laws is not great for small differences between p_1 and p_2 . But the discussion of the two laws and the comparison of the results with those obtained experimentally by Vieille show that the latter gives results much more consistent — in fact quite satisfactorily consistent — with the facts of observations now at our command.

The existence of a wave of the first order gives immediate rise to two waves of the second order, one propagated forward and the other propagated backward toward the piston from which it will be reflected. The result of the reflection of waves from the piston and of the intersections of waves moving toward the piston with those issuing from it gives rise to a whole series of rapidly changing states of the gas. Hugoniot assumed that the state obtained in the gas by moving the piston gradually from the velocity $v = 0$ to the velocity $v = V$ would, owing to the repeated reflections and intersections, approach as its limit the state obtained by increasing the velocity from $v = 0$ to $v = V$ in a sudden impulse. Hadamard proves that at least in some cases this is true. The difficulty of the analysis seems to be such as to prevent a proof of the fact in general.

In the fifth chapter, of only 16 pages, the author passes to the consideration of motions in three dimensions. Here the analytic difficulties are so great that the real treatment of the subject has to be postponed until these are removed in the last chapter. Certain general results, however, are obtainable. In a gas every wave of the second order must be longitudinal with the usual velocity of propagation $(dp/d\rho)^{1/2}$. The transversal discontinuities which exist are stationary. In liquids all discontinuities are stationary and transversal. The author makes use of this fact to show that the waves of the second order do not affect the circulation of the liquid and consequently can

have no influence upon the vortices in the fluid — a result of prime importance in connection with the theory of von Helmholtz. In Note II at the end of the book the subject is continued and the paradox of d'Alembert — that a body may move perpendicular to one of its planes of symmetry without experiencing any resistance — is treated. In Note III the author further shows that waves of the first order do influence the vortices and may either generate or destroy them.

The twenty-two pages of chapter VI are devoted to the study of waves in elastic bodies. In case of small vibrations in an isotropic body if the equations of motions be written in the form

$$\rho \frac{d^2 \mathbf{r}}{dt^2} = M \nabla \cdot \nabla \mathbf{r} + (L + M) \nabla \nabla \cdot \mathbf{r} + \rho \mathbf{F}$$

we find a longitudinal wave of the second order propagated with the velocity of $\theta^2 = (2M + L)/\rho$ and a transversal wave of which the velocity is $\theta^2 = M/\rho$. The first causes no change in the curl and the second none in the density. The results are immediately extensible to non-isotropic bodies such as are encountered in the theory of the propagation of light in crystalline media. In the case that the medium undergoes finite distortions we find that one direction of wave front is capable of propagating three directions of vibration which are mutually perpendicular in the distorted medium. The waves are in general neither longitudinal nor transversal.

The seventh and last chapter is the longest in the book. It contains 87 pages and deals with the theory of systems of partial differential equations and their application to the problem in hand. It was Beudon, a very promising young French mathematician now dead, who in 1897 generalized the theory of characteristics to the case of a system of partial differential equations in any number of variables. In his last chapter Hadamard proposes to develop and fill in this theory and give its application to the propagation of waves. Let us state the problem in the restricted form in which we use it. Given a system of three partial differential equations of the second order in the unknowns ξ, η, ζ and the n variables x_1, x_2, \dots, x_n . The generalization of the problem of Cauchy is this: To determine the functions ξ, η, ζ having given the values of these functions and those of their first derivatives $\partial \xi / \partial x_n, \partial \eta / \partial x_n, \partial \zeta / \partial x_n$ with respect to x_n on a surface M_{n-1} of $n - 1$ dimensions

in the space of n dimensions defined by the variables x_1, x_2, \dots, x_n . The values of the second and higher derivatives may then be computed from the differential equations unless the surface M_{n-1} satisfies a partial differential equation of the first order and sixth degree, $H=0$. All surfaces which satisfy this equation $H=0$ are called characteristics of the given system of equations. This equation itself has characteristics which are lines M_1 defined by the system of equations.

$$\frac{dx_1}{\partial H/\partial P_1} = \frac{dx_2}{\partial H/\partial P_2} = \dots = \frac{dx_{n-1}}{\partial H/\partial P_{n-1}},$$

where P_1, P_2, \dots, P_{n-1} are the partial derivatives of x_n with respect to x_1, x_2, \dots, x_{n-1} taken along the surface M_{n-1} . These lines are called bicharacteristics of the original system of equations. They have the fundamental physical interpretation of corresponding to *rays*. As this review has already reached such length, we cannot go into the details of Hadamard's treatment of differential equations, the problem of Cauchy, and the establishment of the theorems of existence. We may, however, mention that from the physical side, which seems always to be the guiding line in his work, he touches on refraction, Fresnel's wave surface, double and conical refraction, Huyghens's construction, and finally diffraction.

One who does not watch the signs of the times might be tempted to say of this work as Gibbs once said of his Statistical Mechanics that "he was afraid that it was too mathematical for the physicist and too physical for the mathematician." If this be at all true, the fault lies almost wholly with the mathematician who, under the influence of men like Weierstrass and Kronecker and other great pure mathematicians of the past sixty years, has been getting himself as far away as possible from the realm of applications be they never so abstract. A hundred years ago there was not much difference between the mathematician and the physicist. Gauss, Lagrange, Laplace, Poisson, Cauchy, all were known for their researches in theoretical applied mathematics. Of late, under the influence of Klein, there has been a decided trend in Germany toward those problems which concern the applications. The English have seldom turned their attention to other than practical mathematics. The French have always been known for the accuracy with which they pursued a middle course, inclining neither to too bald ap-

plication nor to extreme fantasy in theory. It is therefore not surprising that one of the genius of Hadamard should accomplish some of his best work in this intermediate field. We hope and we believe that such example will influence mathematicians in general to turn their attention more to mechanics, light, elasticity, and the like, instead of confining themselves so exclusively to the pure theory only. So well has the author combined his developments of differential equations and of wave theory that the one will force mathematicians to take an interest in the work while the other cannot longer be neglected by physicists.

One of the most remarkable things about the book is its apparent simplicity and naturalness. When it is all done and read, one is half inclined to feel "I could have done that myself" — but he could not. This is one of the distinguishing features of Hadamard's genius. For it and for the book we may congratulate ourselves on the fact that the author has commenced to publish his courses at the Collège de France. May he soon give us his Calculus of Variations.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
December 23, 1903.

BURKHARDT'S THEORY OF FUNCTIONS.

Funktionentheoretische Vorlesungen. Von H. BURKHARDT.
Bd. I₁, *Algebraische Analysis*, xii + 195 pp. Bd. I₂, *Einführung in die Theorie der Analytischen Funktionen einer Komplexen Veränderlichen*, Zweite Auflage, xii + 227 pp. Leipzig, Veit & Comp., 1903.

IN 1897 appeared Burkhardt's book with the same title as the present revision I₂, and in 1899 appeared II, entitled *Elliptische Funktionen*. As the volume of 1897 was subjected to a thorough review by Professor Böcher in the *BULLETIN*, volume 5 (1899), pages 181–185, it will be necessary to indicate here only the changes that have been made in the second edition, chiefly due to the transfer to I₁ of the more elementary parts on real variables.

Chapter III of the first edition may be characterized as a twelve page catalogue of definitions and theorems (usually without proof) on real variables. Those relating to double