plication nor to extreme fantasy in theory. It is therefore not surprising that one of the genius of Hadamard should accomplish some of his best work in this intermediate field. We hope and we believe that such example will influence mathematicians in general to turn their attention more to mechanics, light, elasticity, and the like, instead of confining themselves so exclusively to the pure theory only. So well has the author combined his developments of differential equations and of wave theory that the one will force mathematicians to take an interest in the work while the other cannot longer be neglected by physicists.

One of the most remarkable things about the book is its apparent simplicity and naturalness. When it is all done and read, one is half inclined to feel "I could have done that myself" — but he could not. This is one of the distinguishing features of Hadamard's genius. For it and for the book we may congratulate ourselves on the fact that the author has commenced to publish his courses at the Collège de France. May he soon give us his Calculus of Variations.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
December 23, 1903.

BURKHARDT'S THEORY OF FUNCTIONS.

Funktionentheoretische Vorlesungen. Von H. BURKHARDT. Bd. I_1, *Algebraische Analysis,* xii + 195 pp. Bd. I_2, *Einführung in die Theorie der Analytischen Funktionen einer Komplexen Veränderlichen*, Zweite Auflage, xii + 227 pp. Leipzig, Veit & Comp., 1903.

IN 1897 appeared Burkhardt's book with the same title as the present revision I_2, and in 1899 appeared II, entitled Elliptische Funktionen. As the volume of 1897 was subjected to a thorough review by Professor Bôcher in the BULLETIN, volume 5 (1899), pages 181–185, it will be necessary to indicate here only the changes that have been made in the second edition, chiefly due to the transfer to I_1 of the more elementary parts on real variables.

Chapter III of the first edition may be characterized as a twelve page catalogue of definitions and theorems (usually without proof) on real variables. Those relating to double integrals are entirely omitted in the new books, being no longer necessary in establishing the lemmas used in the proof of Cauchy's theorem according to the procedure made possible by the recent investigations of Pringsheim, Goursat, and Moore. Of the remaining theorems in the catalogue, the more elementary are established in I_1, while the others are proved in I_2. In the new Chapter III we find treated point sets (Punktmengen), their points of accumulation (Häufungspunkte), continuity of functions of one or two real variables, upper and lower integrals of $F(x)$, conditions for term by term integration or differentiation of a series, and curvilinear integrals.

Theorem VIII on page 92 of the earlier edition, the statement of which was incorrect as far as the use of the term Bereich for Flächenstück U, and the proof of which was wholly inadequate, appears in the new edition in a new setting, with careful proof (pages 83–85, 87, 120) based on the jacobian of two functions of two real variables.

In other respects, we find only very slight changes. A geometric proof is inserted on page 25; a small addition to the section on automorphic function occurs on page 70. Certain confusing errata in the old edition in Fig. 24 and formulas (3), page 136 and formula (13), page 143, have been corrected in the new.

In his Algebraische Analysis, Burkhardt has added an entirely new volume to his series. Approaching more nearly Tannery's Introduction à la théorie des fonctions d'une variable than perhaps any other text, it may however be regarded as unique, at least from the pedagogical standpoint. The book presents, on the one hand, the general theory of irrational numbers and limit processes, and on the other hand the special analytic representation of the elementary transcendental functions. The author believes that the latter without the former would not be satisfactory, while the former without the latter would prove too abstract and sterile. For the comprehension and appreciation of a large part of Burkhardt's Analysis, not a little power of abstraction is demanded of the reader. The author himself remarks that he regards the study of such a book as profitable only when by an earlier acquaintance with the calculus the need has awakened for a purely arithmetic foundation.

The Introduction is concerned with the notion and need in geometry of an extension of the number system beyond that of positive integers, with grounds for the preference of a purely arithmetic to a geometric basis. Chapter I deals with the elementary operations on positive integers, the concept of which is presupposed. For reference, a set of axioms of equality and inequality is formulated. The nature of the chapter may be indicated by the list of the five "fundamental properties of addition": (1) unrestricted applicability; (2) uniqueness of sum; (3) addition is associative; (4) is commutative; (5) is monotonic (from $a > b$ follows $a + c > b + c$ for every c). From these may be derived by formal logic the various "derived properties" of addition. Raising to a power (Potenzierung) is said to be associative since $(a^m)^n = a^{mn}$. But the second member introduces another operation (multiplication). It would seem more natural to deny the associative property since $(a^m)^n \neq a^{m^n}$, in general. The chapter concludes with three pages on the binominal theorem.

In Chapter II, zero and the negative integers are introduced. This is done quite abstractly by regarding numbers as representatives of operations, not of things. A symbol (5) is given to the operation of adding 5 (to an arbitrary positive integer M). These addition operations may be added or subtracted. Thus $(5) + (7) = (12)$, $(7) - (5) = (2)$. Moreover, the symbol $(5) - (7)$ may be interpreted (still in the realm of positive integers) as the operation of subtracting 2. In the latter case, the author fails to remark (as done on pp. 30–31 for the corresponding developments on multiplication and division), that there is a restriction on the "M," a limitation which, however, does not affect the existence of the operation as such. Finally, $(a) - (a)$, written (0), is the identity-operation (Nichtsthun). *In the domain of these operations* (otherwise than in the domain of positive integers), *subtraction may always be performed.* So far, a letter c denotes a positive integer and (c) the addition of c. We next regard a letter in parenthesis as a symbol which, according to circumstances, may denote an addition, or a subtraction, or the identity-operator, and thus make a generalized definition of $(a) + (b)$. This addition is shown to have the above five properties, if for the monotonic property we agree that $(a) > (b)$ or $(a) < (b)$ according as $(a) - (b)$ is an addition or a subtraction. We may therefore exhibit the reckoning with additions and subtractions also as a reckoning with numbers. For this purpose we need two series of numbers, the one, 1_a, 2_a, $\cdots$, for additions, and the other, 1_s, $2_s \cdots$, for subtractions, and finally a sign for the operation (0). As a simplification, readily justified, we denote the first series by $1, 2, \cdots$, and the second by $-1, -2, \cdots$.

There is given a geometric meaning for zero and the negative numbers by means of directed line segments, also by angles with sense. As preparatory to the explanation of a geometric meaning for multiplication of positive or negative numbers, a very useful convention of signs is made for areas of triangles and for volumes of tetrahedra. There appears a contradiction in sign between the final two paragraphs on page 29.

In the thirty pages of Chapters IV and V on integral rational functions and the solution by determinants of 2 or 3 linear equations, there is given the usual development found in elementary algebras. An exception may perhaps be made in respect to the sections on interpolation formulas.*

Chapter VI gives an ideal introduction to the definition of irrational numbers by Dekekind's cut, with simple illustrations leading up to the general developments and with subsequent detailed numerical examples. In this exposition, it is a *theorem* that a convergent number sequence converges to a definite limit; whereas it is a *definition* in G. Cantor's theory of irrationals. In its proof there is slight oversight at the bottom of page 79. With α irrational and ϵ an arbitrary positive number, it is stated that $\alpha - \epsilon$ is an a (rational). In correction, it suffices to note that there is an a_1 such that $a_1 > \alpha - \epsilon$, and hence an index k such that every $c_{k+p} > a_1 > \alpha - \epsilon$. A similar correction is to be made at the top of page 87. Likewise in § 50, even if we make the unnecessary restriction that a be rational; there is an infinitude of rational numbers expressible in neither of the forms a^b, a^B; in correction, one puts into class c all rational numbers $\gtreqless$ any a^b, etc. On page 79, fifth line from bottom, $>$ should be $<$. There are two evident misprints in (4) on page 84, and one in the first line of § 50. In (13), page 95, there should be added, as second restriction, $1 + \delta > 0$. On page 95, fifth line from bottom, read a^m for a^{m+n}. On page 96, after first equality in sixth line, read $-n$ for n; while in (6) read $<$ for $>$.

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* Lagrange's formula is given in its fullest form (page 47) with no mention of the compact form

$$y = \sum_{i=0}^{n} \frac{y_i f(x)}{(x - x_i) f'(x)}, \quad f(x) \equiv (x - x_0)(x - x_1) \cdots (x - x_n).$$

In exemplification of the practical value of the δ, ϵ notation, one may refer to the simple discussion on page 100 of the relative accuracy required in each step for the computation of a root of an irrational number with an assigned accuracy. From the general result it readily follows that, in the extraction of square, cube or fourth roots, one needs the number under the radical only to the same number of decimal places as the desired root.

On page 103, the signs of $-\mu$ and $+\mu$ at the end of (8) and (9) should be interchanged. At the top of page 105, one should have $M > \beta$, $M < b_n$; the developments, moreover, seem to need some minor alterations when $\beta < 0$. Near the bottom of page 137, $f(x) - g$ should replace $f(x)$. Before (5) on page 153, read $z = n/x$.

The elements of purely analytic trigonometry are developed on pages 160–167, thereby making the extension (in I_2) to complex arguments very simple. On page 161, m should be restricted to integral values.

In noting so many errata, the reviewer does not wish to give the impression that the book was either written or printed carelessly. It is only just to state that the list is not merely the result of a reviewer's perusal, but rather of a detailed study for class use, also including various observations on the part of its members. It is thus hoped that the list may prove of use, particularly to those who read only portions of the text.

As the book of 1897 proved so popular, in spite of its bugbear in Chapter III, it requires no prophet to foresee the reception awaiting the present three-volume series of Burkhardt's Vorlesungen.

L. E. DICKSON.

NOTES.

THE fifteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Northwestern University, Evanston, Ill., on Saturday April 2. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor THOMAS F. HOLGATE, 617 Hamline Street, Evanston, Ill., not later than March 10.