

a pamphlet on Vector Analysis printed in 1884, but not published, Gibbs suggested as possible and perhaps preferable the notation  $d/dr$  in place of  $\nabla$ . This idea he never developed, at least so far as is known, and consequently Dr. Fischer's monograph fills an evident gap in the theory.

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### THE MATHEMATICS OF INSURANCE.

*Versicherungsmathematik.* Von ALFRED LOEWY. Leipzig, Sammlung Göschen, 1903. 145 pp.

IN publishing this book the "Sammlung Göschen" has certainly followed out successfully its expressed policy of giving to the public a brief, yet clear and up-to-date development of one of the most interesting applications of mathematical theory. While a reader who is unacquainted with the subject of life insurance would find Professor Loewy's exposition somewhat too condensed, anyone with a knowledge of elementary algebra who has some acquaintance with the business aspect of the subject cannot fail to appreciate the value of this little pocket edition which contains in its 145 pages the development of all the important formulæ needed by the actuary.

While one recognizes at once the meanings of many of the words such as *Nettoprämie* = net premium, *Sterblichkeitstafel* = mortality table, the significance of some of the German expressions, of which a glossary of 15 follows, is not at all evident. Indeed, a few are not to be found in the average German-English dictionary and their meaning can only be learned from the context.

|                                 |                             |
|---------------------------------|-----------------------------|
| Zinsfuß = rate of interest,     | Rückversicherung = rein-    |
| Zinseszins = compound interest, | surance,                    |
| Barwert = present value,        | Bruttoprämie = gross pre-   |
| Leibrente = annuity,            | mium,                       |
| Erlebensversicherung = en-      | Prämienrückgewähr = return  |
| dowment,                        | (of part or whole) of prem- |
| postnumerando = payable sub-    | ium,                        |
| sequently,                      | Rückkaufpreis = surrender   |
| pränumerando = payable in ad-   | value,                      |
| vance,                          | Passiva = liabilities,      |
|                                 | Aktiva = assets.            |

Karenzzeit = period of deferred insurance (a policy payable at death, provided death is not within  $m$  years, is spoken of as a policy with  $m$ -jähriger Karenzzeit).

Zillmersche Prämienreserve [zillmern (verb)] = reserves calculated according to the formula

$${}_m V'_x = \left( P_{(m)} - P_x - \frac{\delta_x}{a} \right) a_{(m)},$$

where  ${}_m V'_x$  is the reserve on the sum 1 after  $m$  years, the insured being of age  $x$  at the time the policy is written;  $P_{(m)}$  = net yearly premium on the sum 1 which an individual, age  $x$ , would have to pay, the term of payments being that which remains for the individual of age  $x + m$ ;  $P_x$  = net yearly premium beginning at age  $x$  and for the sum 1;  $a$  = the present value of a life annuity of sum 1 payable at once;  $a_{(m)}$  bears the relation to  $a$  that  $P_{(m)}$  does to  $P_x$ .

Every time a conventional symbol is introduced its number (Roman) is printed in the margin, thus enabling the reader to find easily the meaning of the symbols in any formula under consideration. As might be expected, the notation is not that of the Actuarial Society of America; the author follows, with some exceptions, the universal standard notation of the International Congress of Actuaries of 1895.

The symbol  $m_x$  is ordinarily used to express the "central death rate"  $d_x/l_x \times \frac{1}{2}$ ; in the present work  $m_x$  denotes the number of deaths out of  $A_x + B_x - C_x$  individuals where  $A_x$  is the number, age  $x$ , under observation,  $B_x$  the number (of the same age) which is added during the year,  $C_x$  the number which is lost track of during the year.

Unlike the English writers, the Germans give special names to the present value of the unit sum  $V = 1/(1+i)$  ( $i$  = rate of interest), and to the expression  $C_x = d_x v^{x+1}$ ; the former is called the "Diskontierungsor Abzinsungsfaktor," the latter the "Zahl der diskontierten Toten des Alters  $x$ ."

The fallacy involved in assessment life insurance is now well understood and Loewy therefore dismisses this subject with a brief and unfavorable mention.

On page 92 it is stated that a well known German company computes the gross premium  $A'_x$  of a one payment life policy by means of the formula  $A'_x = \frac{2}{3} \frac{1}{0} A_x$  where  $A_x$  = net premium. In America the "loading" is higher than this, but on the other hand policies are "participating" and draw dividends.

In the case of entire life policies, yearly payments, the formula is  $P'_x = 1.24 P_x$  for ages exceeding 34 years, making a loading of 24 %. In America the loading is sometimes less and sometimes more than this, varying with the different companies. For non-participating policies it is much less. It is not calculated as a percentage of the net premium however.

In the case of half yearly payments, the Germans add one per cent of yearly net premium ; in case of quarterly payments two per cent is added.

The American law requires an insurance company to have on hand at the end of the first year, when the expenses are heaviest, the full reserve according to a mortality table in which no account is taken of the gain due to selection by medical examination. The large and wealthy companies meet this requirement by a bookkeeping device, transferring from the surplus to the first year's reserve enough to meet the legal requirements. The smaller companies are compelled, however, to resort to the following plan. If the contract is for a twenty payment life policy, they issue a one year term policy, the regular insurance commencing one year later and extending over nineteen years. Actuaries are divided in opinion at present on the moral point involved in this device ; although unjust from the technical point of view it is difficult to see how a new company could come into existence in any other way.

The German law is more flexible, allowing the company to draw upon the first year's reserve, for expenses, to the extent of 12.5 M. per thousand marks of insurance.

Experience shows that the Gompertz-Makeham formula gives the most satisfactory way of graduating mortality tables and it is the method in general use at the present time. For this reason the reviewer regards the author's treatment as too brief and condensed. The mere statement of the final result in the integrated form with the specification of the value of the constants is not sufficient for one who is reading it for the first time.

This fundamental formula may be developed in an elementary way as follows. Suppose that those who die between the ages  $x$  and  $x + 1$  are immediately replaced by other individuals of the same age. The number of deaths during the year under these circumstances, divided by the number  $l_x$  who enter upon this year, is called the "force of mortality"  $\mu_x$  — in the language of the calculus

$$\mu_x = -\frac{1}{l_x} \frac{dl_x}{dx} = -\frac{d \log l_x}{dx}.$$

According to the Institute of Actuaries textbook

$$\mu_x = \frac{d_{x-1} + d_x}{2 l_x} = \frac{l_{x-1} - l_{x+1}}{2 l_x}$$

very closely.  $\mu_x$  can be found from the census statistics (which give the central death rate  $m_x$ ) by means of the relation  $m_x = \mu_{x+1/2}$ .

In 1825 Benjamin Gompertz proposed the formula  $\mu_x = Bc^x$  on the assumption that with increasing age there is an inability, increasing in geometric progression, to withstand destruction. Now death may also be due to chance without previous deterioration. Taking this into account Makeham, in 1860, gave the modified formula  $\mu_x = A + Bc^x$ . From this it follows that  $-d \log l_x / dx = A + Bc^x$ . Integrating, and putting  $-A = \log s$ ,  $-B / \log c = \log g$ , there results  $l_x = k s^x g^{c^x}$  where  $\log k$  is the constant of integration. The values of the constants differ according to the mortality tables from which they are determined. Thus, Loewy gives  $\log s = -0.002527627$ , while the Institute of Actuaries text-book gives  $\log s = -0.002689327$ .

A very interesting feature of the book is the insight given by it into the conditions of German life which help one to understand why deferred temporary annuities and endowments are more popular in Germany than in America. The former are often purchased for the purpose of defraying the expenses of a son at the university. The latter, besides being a provision for old age, are also purchased for the purpose of providing a daughter with her "Aussteuer" and a son with the expenses of the one year of military service. According to the German law those who complete the "Unter-sekunda" and are willing to defray their own expenses are exempted from one of the required two years of service in the infantry.

The book seems to be free from misprints and other errors. By careful condensation the author has reduced it to pocket size. It will be found a very useful little handbook.

SAUL EPSTEEN.