THE HEIDELBERG CONGRESS: SECTIONAL MEETINGS.

(*Continued from page* 217.)

Section IV. *Applied Mathematics.*

Presidents: Klein, Runge, and Hauck. Chairmen: Van Vleck, Volterra, Hadamard, Börsch, and Finsterwalder.

(1) KLEIN (Göttingen): "Ueber die Probleme der angewandten Mathematik, besonders über die pädagogische Seite." †

(2) DELAUNAY (Warsaw): "Ueber das Problem der drei Körper." †

(3) LEVI-CIVITA (Padua): "Ueber das Problem der drei Körper." †

(4) WEINGARTEN (Freiburg): "Ueber einen Fall der Bewegung einer schweren Flüssigkeit mit freier Oberfläche."

(5) VOLTERRA (Rome): "Théorie des ondes."

(6) HADAMARD (Paris): "Sur les équations aux dérivées partielles de la physique mathématique." †

(7) SOMMERFELD (Aachen): "Mechanik der Elektronen." †

(8) GENESE (Aberyswith): "On the development of the Ausdehnungslehre by the principles of statics." †

(9) WEBER (Strassburg): "Einige Bemerkungen zum Vortrag Hadamards." †

(10) ANDRADE (Besançon): "Chronometrische Untersuchungen."

(11) BÖRSCH (Potsdam): "Heutige Kenntnis der Erdgestalt."

(12) FINSTERWALDER (Munich): "Flüchtige Aufnahmen mittels Photogrammetrie." †

(13) PRANDTL (Hanover): "Flüssigkeitsbewegung mit kleiner Reibung." †

(14) KEMPE (Rotterdam): "Ein Gelenkmechanismus." †

(15) RUNGE (Hanover): "Leibniz' Rechenmaschine." †

(16) DISTELI (Strassburg), SCHILLING (Danzig †), and WIENER (Darmstadt): "Communications and demonstrations on models collected in connection with the Congress."

1. In his opening address Klein said that if we compare mathematics to a fortress, the different branches of applied mathematics are represented by the outworks surrounding the

fortress on all sides and connecting it with the outer world. Common to all branches of applied mathematics is then only this, that the mathematical idea in them connects itself necessarily and inseparably with other lines of human interest. Applied mathematics stands in express contrast to those branches of our science which may be considered the citadel of the fortress, to formal mathematics (in the sense of Leibniz), that is to that treatment of mathematical questions which so far as possible ignores any concrete significance of the quantities and symbols employed and concerns itself only with the laws according to which the quantities and symbols may be combined. The welfare of the science requires unquestionably the free development of all its parts. Applied mathematics here undertakes the double rôle of ever conducting to the center from without the stimulus of new problems and conversely of securing for the product of the center due external efficiency. The significance of mathematics within the wide domain of human interest depends therefore on the active vitality of applied mathematics. Hence there where most abundant opportunity is given for the influence of mathematics on wider circles — that is, in the instruction of the youth — we must bring the teaching of applied mathematics to special prominence. From this point of view may be mentioned a recent paper (presented to section VI), "Ueber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen." The object of this paper is to exhibit the theory of functions in such manner and with such illustration chosen from mechanics and physics as will warrant its introduction (as the centralizing agent for all of the more difficult arithmetical questions) into the curriculum of the higher schools. The paper is but a part of a collection soon to be printed on this general subject by a number of well known scientists.

2. When three masses m_1, m_2, m_3 move under the Newtonian law of attraction, the resultant forces pass through the same point which Delaunay calls the center of attraction. This is situated at the center of gravity of fictive masses $m_1\alpha^3$, $m_2\beta^3$, $m_3\gamma^3$ placed at the points m_1, m_2, m_3 (α, β, γ are the sides of the triangle m_1, m_2, m_3). If the coordinates of the masses are x_i, y_i, z_i ($i = 1, 2, 3$), those of the center of attraction may be expressed by

$$x = \Sigma ma^3x/\Sigma ma^3 = \xi/\omega, \quad y = \Sigma ma^3y/\Sigma ma^3 = \eta/\omega,$$
$$z = \Sigma ma^3z/\Sigma ma^3 = \zeta/\omega.$$

The differential equations take the form

$$(x - x_1)\omega = \beta^3\gamma^3\frac{d^2x_1}{dt^2}, \quad (y - y_1)\omega = \beta^3\gamma^3\frac{d^2y_1}{dt^2},$$
$$(z - z_1)\omega = \beta^3\gamma^3\frac{d^2z_1}{dt^2},$$

and similar equations obtained by permuting 1, 2, 3 and α, β, γ cyclically. These equations can be utilized in seeking the particular solutions of the problem.

3. This is the summary of a memoir shortly to appear in the *Acta Mathematica.* The main results which are there announced and discussed are the two following : (P denotes the body of negligeable mass; S, T the two centers rotating uniformly.) 1° Despite the Newtonian law which makes the force infinite at the points S and T, it is possible by elementary transformations to remove all analytic singularities and to represent analytically all the arcs of the path, which fall within suitable regions about S and T. 2° In many cases a simple inspection of the initial conditions enables us to foresee that the distances of P from S and T will remain within fixed limits. The question of stability as stated by Poincaré is therefore answered by Levi-Civita in the affirmative.

6. For a long time, only two sorts of questions were proposed in connection with the determination of an integral of a linear partial differential equation — the problem of Dirichlet or the analogous problem of Bjerknes and Neumann for Laplace's equation ; and the problem of Cauchy for equations with real characteristics. These two classes of problems are far from being the only ones or even the most important ones which come up in applications to mechanics and physics. Indeed in general, whether it be the hyperbolic or elliptic case, one has to do with *mixed* problems, that is to say the boundary is divided into several regions in which the initial conditions are of different kinds. These *mixed* problems are in particular of two sorts. In the elliptic case there is given either the value of the unknown function or that of one of its derivatives

according to the regions considered. In the hyperbolic case there is given the value of the unknown function and of its derivative (as in the problem of Cauchy) on a part of the boundary and the value of the unknown function on the rest. This last mixed problem is fundamentally different from that of Cauchy and from various points of view resembles that of Dirichlet and the other problem of the elliptic case. It depends in an essential way upon the forms of the boundary — a thing which does not happen in the problem of Cauchy.

7. On the basis of the simple and rigorous presentation of the field of a moving electron which Sommerfeld published in the *Göttinger Nachrichten*, 1904, the force and moment opposing the motion can be calculated in a general manner. When these forces are not equal to zero, they give the law of motion. In particular the rotations around a fixed center were discussed. The law of motion appears here as an integral equation and can be transformed into a differential equation of infinite order. There is an infinite series of free fundamental vibrations. In the case of superficial charges these are undamped and are determined by the equation $\tan \gamma = \gamma$. The energy pulsates freely in the interior of the electron and is latent relative to the exterior. If it be imagined that the atom of radium carries out such free vibrations and that its energy of rotation is in some way transformed into energy of translation, then an angular amplitude of small size suffices to explain the observed velocities of the β-rays of radium.

8. This communication forms a sequel to the paper which was read in 1894 at the meeting of the French Association and in which progressive and regressive multiplications were explained by statics for the plane. An idea of the method may be obtained from the following example. Let Σf denote a series of forces in equilibrium and let their rotors be cut by any plane in the points ΣA. The force f at A may be resolved into n normal to and t in the plane. Then ΣnA is a system of masses in equilibrium. Hence the product of a line and a plane may be taken as a point with an easily defined result. One simple geometric result may be noticed. Let four generators of the same system on a hyperboloid of one sheet be cut in the points A, B, C, D by one of a system of parallel planes. Then the *ratios* of the areas of the triangles ABC, BCD, etc.,

are constant, although the configuration does not remain similar to itself.

9. Weber's first note was concerned with the theory of vibrating strings and sought to explain the fact that although the variables enter in a symmetric manner in the equation

$$\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = 0,$$

yet the boundary conditions for $x = 0$ and for $y = 0$ behave differently. (Riemann-Weber, Die partiellen Differentialgleichungen II, §90.) The second note considered Riemann's theory of finite vibrations in air and discussed the question of the connection of the discontinuities discovered by Riemann with the principle of conservation of energy. (*loc. cit.*, §§179–180.)

12. Finsterwalder spoke of exposures in which the photographic viewing-points were not accurately determined. Two exposures in which the constants of the apparatus were known, and which were properly oriented toward the vertical, suffice for reconstructing to scale the object photographed and the viewing-points. The rigorous solution of the problem leads to an equation of the sixth degree for which the speaker gave convenient graphical methods of approximation. To carry out the solution an auxiliary projection of the object was introduced. In this the net of n viewing-points in space is represented by a net of $n(n-1)/2$ points in the plane, lying three by three on $n(n-1)(n-2)/6$ straight lines. In practical work it is better to replace this net by a polar reciprocal net of lines. The method was tried on a practical example in which from eight exposures at six unknown viewing-points the alpine end of the Val di Genova was reconstructed on a scale of 1: 10,000.

13. Although in the motions of fluids arising in technical work the friction in the interior of the fluid plays a very small rôle, the theory of frictionless fluids agrees but poorly with experiment. If a small instead of zero value of the coefficient of friction be assumed its effect becomes noticeable when without it a finite difference of velocity would arise — as at the wall of a rigid body. It is possible to approximate very closely to the actual condition, if the effect of friction be taken account of to the first order in a thin layer along the wall, while in the free fluid no friction be granted. The appearance of surfaces of separation (Helmholtz's vortical sur-

faces) at continuously curved walls becomes recognized as due to friction. Prandtl showed with the stereopticon some experiments and explained them with the aid of the theory.

14. Kempe's paper was for the purpose of demonstrating a link work invented by him, and shown at the exposition held in connection with the congress. The chief object was to draw continuously the curves which he published in 1898 for dividing any possible angle into $2^n - 1$ and $2^n + 1$ parts, and which once drawn require only *one* transversal to find the $1/(2^n - 1)$ and $1/(2^n + 1)$ parts of any angle placed in the center of the diagram — that is to say, to find the complete division of an angle into any equal parts. (*Mémoires de la Société royale des sciences de Liège*, volume 20, 1898; *Zeitschrift für Mathematik und Physik*, volume 49, 1903.)

15. The calculating machine invented by Leibniz and constructed during his life was shown to the meeting. In essentials it agreed with the modern arithmometer of Thomas. The mechanism for carrying the tens is incomplete to such an extent that several successive operations of this sort cannot be accomplished. Thus in adding 1 to 9999 the machine gives 9900 instead of 10000. This inconvenience Leibniz alleviated by introducing a signal which showed the failure to carry the tens. The operator was thus reminded to carry the tens with another stroke of his finger.

16. Schilling divided his communication into two parts: first the different methods of projection for mathematical instruction; second the manifold ways of using the stereopticon for didactic purposes — both amply illustrated with lantern slides. As for the first, he described the preparation of slides, the marking of lines with colored inks, etc. As regards the second point, he spoke in detail of the different opportunities for using tbe stereopticon for geometry, for references to literature, for collecting of formulas, etc.

Section V. *History.*

Presidents: M. Cantor and Stäckel. Chairmen: Zeuthen, P. Tannery, von Braunmühl, and Loria.

(1) CANTOR (Heidelberg): "Einführung in die Geschichte der Mathematik; Hinweis auf neue Resultate." †

(2) TANNERY, P. (Paris): "Résumé de la correspondance de Florimona, Debeaune et de Descartes."

(3) DICKSTEIN (Warsaw): "Wronski als Mathematiker."

(4) SIMON (Strassburg): "Ueber die Mathematik der Egypter." †

(5) ZEUTHEN (Copenhagen): "Gebrauch und Misbrauch historischer Benennungen in der Mathematik." †

(6) SCHLESINGER (Clausenburg): "Bericht über die Herausgabe der gesammelten Werke von L. Fuchs und Ueberreichung des I. Bandes."

(7) ENESTRÖM (Stockholm): "Welcher Platz gebührt der Geschichte in einer Encyclopädie der mathematischen Wissenschaften?" †

(8) VON BRAUNMÜHL (Munich): "Zur Geschichte der Differentialgleichungen." †

(9) SUTER (Zürich): "Zur Geschichte der Mathematik bei den Indern und Arabern." †

(10) LORIA (Genoa): Sur l'histoire de la géométrie analytique." †

(11) VAILATI (Como): "Intorno al significato della differenza tra gl'assiomi ed i postulati nella geometria greca." †

1. Cantor emphasized the fact that the investigations in the history of mathematics were capable of development on two sides — toward the future, because what a few decades ago was the present is now history — toward the past, because the philologists and archeologists are discovering new material. Examples show how the history of mathematics has thereby been changed, and also how the philologist and archæologist have grounds for taking account of the investigations of mathematicians. Thus the history of mathematics has more and more acquired the rank of a branch of science justified in itself and taught in special courses at many places of higher instruction. To it other branches of knowledge are connected, even a general history of science — toward which the speaker adopted a rather doubting attitude.

4. Simon protested against a point of view adopted even in the second edition of Cantor's great historical work. He held that the papyri could not be wholly trusted as adequately representing the knowledge of the Egyptians. He pointed out that the majority of the papyri should be regarded as pupils'

note-books, often transcribed by persons of moderate attainments at best. This he corroborated with examples. Further examples show that the Egyptians were in possession of a number of mathematical theories, the principles of similarity, the elements of descriptive geometry, etc.

5. As an example of nomenclature which, despite its departure from strict historical accuracy, should be respected as sufficiently descriptive, Zeuthen mentioned "the Pythagorean theorem" and "Napierian logarithms." On the other hand he disapproved of the terminology adopted by some who at present are interested in the foundations of geometry. They apply the term "theorem of Pascal" merely to a special case which in no wise interested Pascal; they use "axiom" in the sense in which Euclid applied the term "postulate"; they call "axiom of Archimedes" a postulate which is probably due to Eudoxus and which at all events was well known to Aristotle and Euclid (V, def. 4). In fact Euclid expressly develops in his first four books that which is at present called "non-archimedean geometry."

7. Eneström first remarked that by the history of mathematics he understood a useful history of the development of the mathematical theory, and gave to the word encyclopedia the same meaning as the founders of the mathematical encyclopedia now appearing. He then investigated the aim of such an encyclopedia and came to the conclusion that not only should historical notices be inserted in the special articles but that there should be added a general exposition of the historical development of the discipline. Next he mentioned what historical questions should find place in such an exposition and emphasized that the arrangement which he spoke of could be regarded really as a programme for an extension of the original plan of the encyclopedia.

8. Newton in his Methodus fluxionum treated differential equations of the form $F(\dot{x}, \dot{y}, \dot{z}, \cdots, x, y, z) = 0$, for which he gave as a sole example $z\dot{x} - \dot{z} + x\dot{y} = 0$. His solution of this so-called pfaffian differential equation shows that he thoroughly understood the character of this sort of equations, and only the lack of a functional notation prevented him from giving the general representation of both integrals. Moreover he evidently

gave only this one example because it was beyond his power to give to the solution a geometric interpretation — Monge (1784) was the first to succeed in so doing. The first *partial* differential equation is found in Euler, and came in his way when he sought the multiplier of the differential equation $dz + Pdx = 0$. From 1762 on Euler and d'Alembert treated partial differential equations systematically. The most general equation solved in this year by Euler is $z = \phi(p, q)$. D'Alembert, however, gave a method for solving the linear partial differential equation of the first order

$$\frac{\partial z}{\partial x} + \phi(x, y)\frac{\partial z}{\partial y} + z\psi(x, y) + \chi(x, y) = 0,$$

and Laplace in 1773, that is a year before Lagrange, integrated the equation

$$\frac{\partial z}{\partial x} + \phi(x, y)\frac{\partial z}{\partial y} + \psi(x, y, z) = 0.$$

9. Suter first treated the two remarkable formulas for polygons found in Bhāskara's Līlāvatī (Cantor, Vorlesungen, 2d edition, volume 1, page 618), and sought to show how the Hindus could have arrived at these formulas. Secondly, he gave grounds to establish the probability that the "liber augmenti et dimensionis," published by Libri in his Histoire des sciences mathématiques en Italie (volume 1, pages 304–368), was written by Abū Kāmil Shoǵā' ben Aslam (al. Abraham), an Arabian mathematician living in Egypt about the year 900.

10. Loria began with an aperçu of Baltzer's Analytic Geometry, and asked how that author could say that this branch of mathematics is a grand-niece of Greek geometry. He remarked that this relationship becomes evident if one turns to the Geometry of Descartes, of which he pointed out the resemblance rather to the works of Apollonius than to the modern treatises on analytic geometry, which resemble much more closely the Isagoge of Fermat. It is known, however, that the ideas of this great man have become appreciated only in recent years, whereas those of Descartes spread and developed in an astonishing manner. Loria specially cited among the continuers of Descartes, Roberval, de la Hire, and the marquis de l'Hôpital, and he then stopped to mention the contributions

which Newton made to this doctrine. The appearance of Euler was the signal for the commencement of a new era in analytic geometry as in the other branches of pure and applied mathematics. Despite this, despite the great work of Cramer, it was Lagrange who was destined to place geometry on a true analytic basis. His ideas, combined with those which rendered Monge immortal, appreciated and completely developed by Lacroix and Biot, finally transformed a chapter on the introduction to infinitesimal geometry into an independant branch of pure mathematics. The period of Lagrange, Monge, and Lacroix has therefore an extraordinary importance comparable to that of the period of Descartes and Fermat. Loria said that it is these periods which should have the especial attention of whosoever would write a history of analytic geometry; and he closed by expressing the hope that mathematical literature might be enriched by such a history, of which the necessity could not but be admitted by all.

11. In a remarkable passage of his commentary on Euclid's elements, Proclus alludes to three different significations which had been attributed by his predecessors to the distinction between axioms and postulates. According to one, the postulates were the propositions affirming that a certain construction can be effected, while the axioms were the propositions (undemonstrated) affirming that a construction, supposed or proved to be possible, could not be effected without the resulting figures enjoying or not enjoying some property or other (which is not implied in its definition). According to the second, the properties (undemonstrated) relative to relations which can subsist between any quantities (geometric or not) are called axioms, while the propositions (undemonstrated) of which the import does not extend beyond the field of geometry or spatial relations are called postulates. According to the third, axioms are those properties, the truth of which follows from the very definition of the terms which figure in it, while postulates are the other undemonstrated propositions which are not in this category and which therefore may be denied without falling into contradiction. It is interesting to note that each of these three distinctions is related to criteria, the importance of which is becoming emphasized by recent researches on the logic of mathematics. The first, in part, is relative to the distinction between particular propositions and general propositions in the

sense of Leibniz. The second is relative to the distinction between pure and applied mathematics as stated by Russell in his Principles of Mathematics. Finally, the third agrees with the distinction which modern philosophy expresses by setting in contrast synthetic and analytic judgments. It is important to bring out the fact that in none of the three distinctions mentioned by Proclus is it a question of any contrast (between axioms and postulates) relative to their different "degrees of evidence."

Section VI. *Pedagogy.*

Presidents: Schubert and Treutlein. Chairmen: Lampe, Greenhill, Fehr, Schotten, and Gubler.

(1) KLEIN (Göttingen): Ueber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen."

(2) SCHUBERT (Hamburg): Elementare Berechnung der Logarithmen."

(3) BUFFA (Nocera): "Primo studio della geometria piana."

(4) GREENHILL (London): "Exercises in practical mathematics."

(5) GREENHILL (London): "Teaching of mathematics by familiar applications on a large scale." †

(6) GUTZMER (Jena): "Ueber die auf die Anwendungen gerichteten Bestrebungen im mathematischen Unterricht an den deutschen Universitäten." †

(7) LORIA (Genoa): "Sur l'enseignement des mathématiques en Italie." †

(8) VERONESE (Padua): "La Laguna di Venezia" (by title).

(9) FEHR (Geneva): "Enquête de la revue internationale, L'Enseignement Mathématique, sur la méthode de travail des mathématiciens." †

(10) STÄCKEL (Kiel): "Ueber die Notwendigkeit regelmässiger Vorlesungen über elementare Mathematik an den Universitäten." †

(11) FRICKE (Braunschweig): "Ueber den mathematischen Unterricht an den technischen Hochschulen in Deutschland." †

(12) ANDRADE (Besançon): "L'enseignment mathématique aux écoles professionnelles et les mathématiques de l'ingénieur."

(13) SCHOTTEN (Halle): "Welche Aufgabe hat der mathematische Unterricht auf den deutschen Schulen und wie passen die Lehrpläne zu dieser Aufgabe?" †

(14) SIMON (Strassburg): "Ueber Unterricht in der sphärischen Trigonometrie." †

(15) THIEME (Posen): "Wirkung der wissenschaftlichen Ergebnisse auf den Unterricht in der elementaren Mathematik."†

(16) SOUREK (Sofia): "Ueber den mathematischen Unterricht in Bulgarien."

(17) MEYER (Königsberg): "Ueber das Wesen mathematischer Beweise."

(18) FINSTERBUSCH (Zwickau): "Ueber eine neue und vor allem einheitliche Methode, die Rauminhalte der Körper zu bestimmen, deren Querschnittsfunktion den dritten Grad der Höhe nicht übersteigt."

(19) BRÜCKNER (Bautzen): "Ueber die diskontinuierlichen und nicht-konvexen gleicheckig-gleichflächigen Polyeder." †

1–4. These papers were distributed in printed form.

5. A collection of exercises was distributed showing the method of instruction in dynamics in the advanced class of the Ordnance College at Woolwich. Attention was directed to the concrete practical nature of the interest, as time does not allow a complete theoretical study. The use of a mathematical workshop was recommended as a useful stimulus. In this the student verifies the laws of energy and momentum with homely experiments. In our present laboratories we are really following much earlier methods. Galileo's swinging lamp, for example, suggested to him not mechanical application of the pendulum, but a simple means of determining equal intervals of time.

6. In the first part of his communication Gutzmer spoke of the rise of the tendency toward applications in university instruction. In the second part he took certain exceptions to the theses of Stäckel (*Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 13, pages 340–1). While agreeing in the main with Stäckel, he wished to emphasize the differences: 1° that every student of mathematics should pursue descriptive geometry, as in southern Germany, 2° that in pure mathematics the requirements for the examination *pro facultate docendi* be not lowered from the present standard, 3° that in applied mathematics more attention be paid to technical mechanics than is recommended by Stäckel, in order that the mathematicians might better understand the problems of technical science and be better prepared for teaching in schools of

technology, 4° that the requirements in applied mathematics and physics for the doctorate be as accepted at Göttingen and Jena. In the third part he took up the pedagogic development of the teacher. The scientific development (even in a pedagogic way) should be the task of the universities, but the introduction to the practice of teaching could best be obtained from experienced school teachers.

7. Loria commenced by remarking that the geometric traditions in Italy were frankly Greek, at all events they were at most partially weakened by the successive governments which followed each other. It was Cremona who proposed and brought to success the reëstablishment of instruction on a good basis by suggesting to the Italian government to force, in the classical schools, a return to the pure sources of Greek geometry. This was moderated after a few years by demanding merely that instruction follow the method and not necessarily the text of Euclid. This movement gave rise to a series of good modern treatises such as those of Sannia, d'Ovidio, and Faifofer; while eminent scientists, as Paolis, Veronese, Enriques, and Amaldi, turned their attention to the foundations of geometry. The results they obtained, the discussions aroused by their propositions, the attempts which their students made to use these in instruction, infused the body of instructors with new life. Of this a striking proof is found in the association Mathesis, allying the instructors of the secondary schools of Italy for the purpose of "making use of the progress of science for the purposes of the schools." Loria next examined two questions: 1° Is it useful from a didactic point of view to abandon the euclidean method and to treat conjointly the geometry of plane and space; 2° Is there any way of increasing the profit which the student should get from secondary instruction? Finally the speaker touched on the Italian technical schools and mentioned the changes which sooner or later must take place in their curricula. (The paper will shortly appear in *l'Enseignement Mathématique.*)

8. Fehr presented the series of questions which the *Enseignement Mathématique* submits to mathematicians to ascertain their method of work. The object of the inquiry is to collect the opinions of mathematicians on the methods of their work in order that from the mass of replies there may be sifted out a

number of points of information and advice which may be useful not only to young mathematicians but to mathematical instruction in general. The inquiry contains thirty questions of a psychologic and physiologic nature. It is in fact interesting to know not merely the habit of work but the general hygienic habits of mathematicians, and the means which they judge best fitted to facilitate intellectual work. It is to be hoped that all mathematicians will lend their collaboration in the form of replies to these questions in order that the results of the inquiry may well represent the general opinion. (The series of questions with blanks for reply may be obtained of Professor H. Fehr, 19 rue Gevray, Geneva, Switzerland.)

10. The practical pedagogic training of the candidates for positions as teachers does not belong to the universities; but it is desirable that the present chasm still existing between university science and school instruction be filled in. Besides vacation and information courses in which the universities are called to coöperate there should be for this purpose in the regular curriculum itself 1) elaborate consideration of the applications, 2) lectures in which elementary mathematics is treated from a higher standpoint. Such lectures on elementary mathematics 1) would be easily fitted in to the present scheme of university instruction and indeed should find their place at its close, 2) would notably aid and support the future pedagogic development of the candidates, 3) would also contribute to keep the teachers of mathematics in living contact with their science. To accomplish this aim these lectures should awake an interest for the elements and give a deeper insight into them; in this the scientific and mathematical, and the historical and literary import alike demand consideration.

11. Fricke spoke of the differences which arose about ten years ago between the teachers of the technical subjects and teachers of mathematics in technological schools. He then entered into opposition with Klein, regarding the latter's proposition to introduce the elements of differential and integral calculus and of the theory of functions into the curricula of the higher secondary schools. In closing he warmly recommended the striving on the part of the technical schools to prepare teachers of mathematics and technology for the gymnasia and secondary schools.

13. Schotten, without at all contesting the formal value of the instruction, wished that the material side be better attended to. He insisted on the difference between the problems given out as exercises and the applications. In a reform, however, we should not pay attention merely to mathematical and physical science; the entire field must be considered. Plans had been proposed on different sides, especially by those interested in hygiene and in German nationalism, for a new arrangement of higher instruction. It seems as if a new ideal of education was coming in. The speaker asked that in outlining the curricula every tendency to overload be abandoned. Rather keep to the motto "hungry better than overfed." The aim of mathematical instruction should be to foster a pleasure in reflective thinking and individual work. To effect this several changes were suggested, the present curricula being unhappily chosen. For example, let the matter taught be collected in separate grades, and let the unfortunate effect of the hurry-to-get-there ("Hetzbetrieb") be removed.

14. Simon first remarked that problems which lead to a quadratic equation may be solved without the introduction of imaginaries, whereas the equation of the third degree requires necessarily imaginary numbers. It is therefore no mere chance that Cardan was the first to work with complex numbers. As a consequence the speaker inferred that the curricula of the secondary schools should either contain no imaginaries or introduce them in connection with the cubic equation. Next he passed to the consideration of spherical trigonometry. He recommended a number of changes in the method of instruction. In short, these amounted to nothing but a return to the method followed by Euler.

15. The scientific investigations which may be of use in elementary mathematics were divided by Thieme into three categories: First, science has given to the elements a logical foundation; second, a certain number of recent researches bring more general questions to the domain of elementary mathematics and thus render possible a deeper insight into the connections between the branches; third, of late years certain parts of elementary mathematics such as the geometry of the triangle have made great advances. In dealing with the question in how far these investigations can be used in the instruction of ele-

mentary mathematics account must be taken of the time alloted to instruction, of the comprehension of the student, of the present state of instruction, and the extent to which these researches have spread among the teachers. Many changes which could be heartily wished for cannot be made owing to the limits imposed by such conditions as those just stated — for instance, the ability to comprehend on the part of the students in the secondary schools may not be sufficient. But in many points the acquisitions of science should be more taken account of than is the case. Science cannot turn over to the beginner, whether in arithmetic or in geometry, a complete and rigorously logical system, instruction must proceed along the way of intuition. But this only within certain limits. The intuitive idea of positive and negative number as profit and loss should be used only in the first grade, not in the second or third. Moreover the conceptions of equal, greater, less, sum, product, power are defined only for positive integers ; if they be used for negative, rational, irrational and imaginary numbers, one should define them for such numbers. Hence the equations

$$(-a)(-b) = +ab, \quad a^{-n} = \frac{1}{a^n},$$

$$a + bi = c + di \text{ if } a = b \text{ and } c = d,$$

should be regarded not as theorems but as definitions. Even in the lower grades the conceptions should be enunciated correctly and their domain of validity given, although their logical development may be allowed to remain in the background. In the upper grades the whole logical development of arithmetical conceptions and laws in a connected manner should be given to the students. Here the irrational and imaginary numbers should be introduced with care. Infinite series and a short introduction to differential and integral calculus should here find a place. In geometry a strict, complete logical system is beyond the student's power of conception, but many things that science has shown to be false may be modified. At first one may leave aside definitions of surfaces, curves, straight lines, and points. All the definitions in use are inaccurate. The majority of the definitions of angle are equally bad. The many inaccurate proofs should be given up. To be sure, in the lower grades we cannot exact the use of the complete systems of axioms as given by Pasch, Veronese, Peano, and Hilbert ; we must be satisfied with the axioms of connection and of parallels. In the upper grades, however, the student should learn what a rigorous and connected logical system geometry really is even if he does not carry the system through in detail. The speaker went on further to explain how and why the elements of projective geometry and nomography should be introduced. He closed with a short résumé of the actual state of instruction in Germany, and cited in behalf of his opinions the Arithmetik of Stolz and Gmeiner, the Encyclopedia of Elementary Mathematics by Weber and Wellstein, and the geometric treatises of Pasch, Hilbert, Veronese, and Ingrami.

19. Brückner, after some introductory remarks on the historical development of the problem of determining the equiangular, equisuperficial polyhedra — a problem solved for convex polyhedra by E. Hess — proceeded to give a résumé of the results he had found concerning the non-convex and discontinuous polyhedra of this type. Beside the sphenoids of the octahedral and icosahedral systems (7 and 5 groups, respectively) he went into the stephanoids of these two systems (3 and 11 combinations, respectively). The models of these discontinuous "null-polyhedra," and those of the continuous non-convex polyhedra with positive content, and those of Möbius's polyhedra of the icosahedral system were exhibited. (The complete results will be published in the *Nova Acta* of the Leopold Academy.)

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THE BRESLAU MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

The annual meeting of the Deutsche Mathematiker-Vereinigung, forming a part, as usual, of the annual meeting of German scientists and physicians, was held this year at Breslau, in Silesia.

The first forenoon, Monday, September 19, was devoted to a general meeting of the entire association. In the afternoon of the same day the sessions of the different sections began. The convention consisted of 13 scientific and 17 medical sections,