let t represent any operator of order p which transforms the generator s of a cyclic group of order p^{α_1+1} into its $p^{\alpha_1}+1$ power. The commutator quotient group of the group generated by s and t is clearly of type $(\alpha_1, 1)$. Let t' represent any operator of order p^{α_2} which is independent of s and t. The operators s and tt' will then generate the required non-abelian group. By forming the direct product of this non-abelian group and some abelian group any additional invariants may be introduced into the commutator quotient group. Hence the theorem is proved.

STANFORD UNIVERSITY,
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NOTE ON ISOTHERMAL CURVES AND ONE-PARAMETER GROUPS OF CONFORMAL TRANSFORMATIONS IN THE PLANE.

BY PROFESSOR C. L. BOUTON.

IN the January number of the BULLETIN, page 180, Mr. J. E. Wright integrated a certain differential equation by determining a continuous group which the equation admits. In solving the problem Mr. Wright determines a group of *conformal* transformations with *given* path curves. In this connection, it is an obvious problem to find the necessary and sufficient conditions under which a conformal group with given path curves shall exist. The solution of this problem is given in the following theorem :

THEOREM. — *A one-parameter group of conformal transformations with given path curves exists when and only when the given curves form an isothermal family.*

Although this theorem seems very obvious the writer cannot find it in print, and, therefore, gives two easy proofs for it.

I. Let $Uf = \xi p + \eta q$ be the symbol of the infinitesimal transformation of the group. Since the group is to be conformal, we must have $\xi + i\eta = \phi(x + iy)$. The differential equation of the path curves is $\eta dx - \xi dy = 0$. From this equation we have

$$\frac{dx + idy}{\xi + i\eta} = \frac{dx - idy}{\xi - i\eta},$$

or

$$\frac{d(x+iy)}{\phi(x+iy)}=\frac{d(x-iy)}{\phi(x-iy)}. \tag{1}$$

Let

$$\int\frac{d(x+iy)}{\phi(x+iy)}=u(x,y)+iv(x,y).$$

We then have, on integrating (1), $v(x, y) =$ constant, which is the equation of the path curves. But $v(x, y)$ is a solution of Laplace's equation, hence the path curves of any conformal group are isothermal curves. Moreover, if any solution of Laplace's equation be given, we may use this as $v(x, y)$ to determine ϕ, and we see at once that ξ and η are uniquely determined, save as to a multiplicative constant. That is, given a family of isothermal curves, there is one and only one group of conformal transformations which has these curves as path curves.

II. Let us assume that the path curves are given as the integral curves of the differential equation $y' = F(x, y)$. Then we must have $\xi p + \eta q \equiv \rho(x, y)[p + Fq]$. Or $\xi \equiv \rho$, $\eta \equiv \rho F$. The conditions $\xi_x = \eta_y$, $\xi_y = -\eta_x$ give

$$\frac{\partial \log \rho}{\partial x}-F\frac{\partial \log \rho}{\partial y}=F_y,$$

$$F\frac{\partial \log \rho}{\partial x}+\frac{\partial \log \rho}{\partial y}=-F_x.$$

Solving these equations, we have

$$\frac{\partial \log \rho}{\partial x}=\frac{F_y-FF_x}{1+F^2},\qquad \frac{\partial \log \rho}{\partial y}=\frac{-F_x-FF_y}{1+F^2}.$$

Applying the condition of integrability to these equations, we find as the necessary and sufficient condition that ρ shall exist

$$(1+F^2)(F_{xx}+F_{yy})-2F(F_x^2+F_y^2)=0. \tag{2}$$

But this is the necessary and sufficient condition that $y' = F(x, y)$ shall have as its integral curves a family of isothermal curves.* If this condition is satisfied the value of ρ is given

* See Lie-Scheffers, Differentialgleichungen, p. 157; Kasner, BULLETIN, vol. 10 (1903–04), p. 342.

by the equation

$$\rho = e^{\int \frac{F_y dx - F_x dy}{1 + F^2}} / \sqrt{1 + F^2},$$

and is determined save as to a multiplicative constant. The group is therefore uniquely determined.

In the second proof it is assumed that the meaning of the condition (2) is known. If, however, the condition (2) had not been derived independently, the two proofs together show that (2) is the necessary and sufficient condition that the integral curves of $y' = F(x, y)$ shall form a system of isothermal curves.

HARVARD UNIVERSITY,
February, 1905.

ARENDT'S DIRICHLET'S DEFINITE INTEGRALS.

G. Lejeune Dirichlet's Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen. Herausgegeben von G. ARENDT. Braunschweig, Vieweg und Sohn, 1904. xxiii + 478 pp.

This book is almost a literal reproduction of the course on definite integrals which Dirichlet gave at Berlin during the summer of 1854. It is not its aim to give any account of the development of the subject during the last fifty years. The book on definite integrals by Meyer * contains discussions of trigonometric series, potential and other matters, taken partly from other courses of Dirichlet, and partly from his own investigations. Whether the new book encroaches on the older one is not necessary to discuss, for Meyer has long been out of print and it is certainly worth while to have the Dirichlet course accessible, essentially in the form in which it was given. Moreover, apart from the questions of continuity, integrability, length, area, uniform convergence, etc., the great body of subject matter is to-day what it was then.

After defining continuity, an integral is discussed by means of a figure which illustrates the area included between two ordinates, the axis of X and a continuous curve. The same problem is then treated analytically, for an arbitrary division of the in-

* Vorlesungen über die Theorie der bestimmten Integrale zwischen reellen Grenzen, mit vorzüglicher Berücksichtigung der von P. Gustav Lejeune-Dirichlet im Sommer 1858 gehaltenen Vorträge über bestimmte Integrale. Von Dr. Phil. Gustav Fěrdinand Meyer. Leipzig, Teubner, 1871.