$$a = \Sigma x_1 i_1, \quad \beta = \Sigma y_1 i_1, \quad \gamma = \Sigma z_1 i_1,$$

$$a\beta\gamma = V_3 . \, a\beta\gamma + V_1 a\beta\gamma, \quad V_3 a\beta\gamma = \Sigma \mid x_1 y_2 z_3 \mid i_1 i_2 i_3,$$

$$V_1 a\beta\gamma = -\Sigma y_1 z_1 \Sigma x_1 i_1 + \Sigma x_1 z_1 \Sigma y_1 i_1 - \Sigma x_1 y_1 \Sigma z_1 i_1,$$

$$a\beta\gamma\delta = V_4 a\beta\gamma\delta + V_2 a\beta\gamma\delta + V_0 \alpha\beta\gamma\delta,$$

and so on for higher products. The different "products" of the Ausdehnungslehre appear at once in their proper places as partial products among all the partial products of one *associative* product. This algebra is of order $2n$, and, if $n = 2m$, is the product of m independent quaternion algebras, if $n = 2m + 1$, is the product of m independent quaternion algebras and the algebra of positives and negatives. Division is possible under much the same restrictions as in a Weierstrass commutative algebra. We also see that the multiple algebra of n-dimensional space is not necessarily a theory of matrices of order n.

The volume is what its author intended it to be, a handy manual for those who desire to learn quaternions and quaternion methods. The appearance of the book is pleasing, and to the reviewer it seems that the simple notation of Hamilton can scarcely be called improved when one views a page full of heavy type, artificial signs, and foreign alphabets. We believe a letter is superior to an arbitrary mark for indicating a process, and indices clearer than fonts of type and sets of alphabets. Professor Joly's text speaks for itself. The book appeals, finally, to the pure mathematician as well as to the physicist.

JAMES BYRNIE SHAW.

SHORTER NOTICES.

Geschichte der Mathematik in XVI. und XVII. Jahrhundert. By H. G. ZEUTHEN, Deutsche Ausgabe von RAPHAEL MEYER. *Abhandlungen zur Geschichte der mathematischen Wissenschaften,* XVII. Heft. Leipzig, B. G. Teubner, 1903. Pp. viii + 434. Price 16 Marks.

AT first thought it may seem strange that Professor Zeuthen should attempt to go over the same ground so recently covered by Cantor in the latter part of volume II and the first part of volume III of the latter's Vorlesungen. A little investigation will show, however, that the field was by no means completely covered by Cantor, and therefore any further contribution to the history of the two most important centuries in the development of modern mathematics must be welcome. No one treatise will ever completely tell the story of the solutions of the general cubic and quartic, of the struggles for recognition of the modern symbolism, of the renaissance of the theory of numbers, of the birth of the analytic geometry, and of the steps leading to the invention of the fluxional and differential forms of the calculus. The field is so extensive that a single writer can hope only to add a little here and there to Cantor, to rearrange and clarify his work, or to give more intensive study to a few details.

Professor Zeuthen's idea is to prepare a work that sets forth the formative period of modern mathematics in such a way that the book may be read by itself, without reference to preceding volumes. This is in harmony with his plans as developed in his other works, although he confesses what is manifest to anyone, that it is imposible to carry it out satisfactorily. It has also been his intention to prepare a book for mathematicans rather than historians, and so there is little attention given to the setting of his facts in relation to the general progress of the race. Mathematical causes, not national development, or humanitarian movements, or the influences of the Renaissance, are made the sole points of departure in the study of mathematical discovery.

That the secondary material has been carefully digested goes without saying, with all who know Professor Zeuthen's earlier writings. That a great many original sources have been examined is probable. It is, however, exceedingly unfortunate that all important sources are left specifically unmentioned. Students who consult a book like this desire, in most cases, to carry their investigations further; they wish a bibliography that is fairly complete in important details; they wish to verify an author's interpretation by referring to the source; but in this particular the work under review offers not the slightest assistance. It therefore fails of what should be one of its most worthy purposes.

The treatise opens with an historical and biographical survey of eighty pages, in which each of the writers whose works are to be discussed is treated with great brevity and without a very careful adjustment of space to the man's relative importance.

For example, Widmann (who spelled his name Widman) is twice mentioned approvingly, although he was of the fifteenth century, while Paciuoli (more properly, if we may trust Staigmüller, Paciuolo), whose works extend into the sixteenth century, fares not even as well as this. And yet Paciuolo was one of the most influential of the early cossic writers, not to speak of his presentation of elementary geometry. Why, too, if Riese deserves a place, should not Köbel, and Trenchant, and a dozen other arithmeticians be named? If Euclid is to be referred to more than twenty times, why should the influence of Nicomachus and Boethius be wholly ignored? There were several editions of the latter's arithmetic in the sixteenth century, and this work had a noteworthy influence in the development of the theory of numbers. And if Euclid is worthy of such attention why should the interesting controversy between Buteo and Finæus on the quadrature problem be wholly neglected? Neither of these men is mentioned, although each was influential and the latter was a rather prolific writer. Of course the reply is that these men are primarily arithmeticians, even though they wrote on other mathematical subjects. But what about Scheubel? He was primarily an algebraist, and a noteworthy one in some respects. And why should the great influence of the fifteenth century Peuerbach be entirely ignored if his pupil Regiomontanus is to have prominent place? Since some discussion is given to figurate numbers, why should a work like that of Schonerus (De Numeris figuratis Lazari Schoneri liber, Frankfort, 1586) be unmentioned, and, on the numerical side, why should such a classical treatise as that of Tonstall not deserve a passing reference? Indeed, the more one looks over the list of names in the historical introduction, the more convinced he becomes that Professor Zeuthen has neither judiciously selected in the field of minor contributors, nor attempted a complete account. The second part of the work, Die Analyse des Endlichen, is somewhat more satisfactory. This part begins with the algebraic solution of the cubic and quartic equations, contributing nothing new, however, to this interesting chapter on algebra. It considers in a rather superficial way the symbolism of the subject. Here again a lack of completeness is apparent at once. A reader might naturally infer that the exponential symbols of Stevin, Bombelli, Bürgi, and Chuquet practically exhausted the list, and that ℞ 2 and $\sqrt{2}$ were the only signs for the square root, and $\sqrt{c}$ and $\sqrt{(3)}$ for cube root.

Cardan's symbols, and Scheubel's, and the interesting use of $L2$ for $\sqrt{2}$ and $2\,L$ for $2\,x$, by Ramus, and various other curious and common forms, are unmentioned. Indeed, the work offers but little help in reading the older authors, and the chapter must accordingly be considered disappointing.

This topic is followed by a review of the general theory of algebraic equations, trigonometry in relation to algebra, numerical computation before the invention of logarithms, the invention of the latter, the number theory before and during the time of Fermat, the theory of combinations and probability, and the projective and analytic geometry of the seventeenth century.

The third and most important section of the work, and that for which it will probably be chiefly consulted, relates to the invention of the calculus. Professor Zeuthen rightly sets forth the important bearing of mechanics upon the early steps in this theory. He shows that applied mathematics demand some form of integration, and that in the efforts of Kepler, Cavalieri, Fermat, Pascal, Wallis, and others, must be sought the first evidences of the modern integral calculus. He then shows that the study of tangents before Newton's time demanded an approach to differentiation, as seen in the labors of Torricelli, Roberval, Descartes, Hudde, Fermat, Huygens, and de Sluse, and, he might well have added, Barrow.

The work closes with a critical study of the respective merits of Newton and Leibnitz in the invention of the calculus, adding nothing that is new to the long drawn out controversy, but agreeing with the general view of the present day as to the priority of the former and the independent work of the latter.

In spite of the incompleteness of various portions of the work, and in spite of the absence of all bibliographical aids to the student, Professor Zeuthen's work will justly rank as a worthy contribution to the history of the two most interesting centuries in the historical development of mathematics.

DAVID EUGENE SMITH.

Introduction à la Théorie des Fonctions d'une Variable. Tome I. By JULES TANNERY. Paris, Hermann, 1904. ix + 419 pp.

THE excellence of Tannery's book of 1886 as an introduction to the theory of functions of a real variable is so generally conceded that it seems sufficient in a notice of the revised edition to point out clearly that the new work in two large volumes is virtually a new treatise, constructed on broader and