GALOIS FIELD TABLES FOR $p^n \leqq 169$.

BY DR. W. H. BUSSEY.

(Read before the American Mathematical Society, September 7, 1905).

EVERY field of a finite number of marks may be represented as a Galois field of order $s = p^n$, where p^n is a power of a prime. The $GF[p^n]$ is defined uniquely by its order, and is therefore independent of the particular irreducible congruence used in its construction. In each of the following tables the $GF[p^n]$ is constructed by means of a primitive irreducible congruence which appears at the top of the table. The marks of each field are arranged in two tables. In each table each mark appears as a power of a primitive root i, and also as a polynomial in i of degree $k \leqq n - 1$. The coefficients in this polynomial are integers reduced modulo p. The mark $Ai^k + Bi^{k-1} + \cdots + Di + E, A \neq 0$ is denoted by $AB \cdots DE$, a symbol consisting of its detached coefficients in order. Zero coefficients must not be omitted. This is the usual symbol for a positive integer in the notation of the number system whose base is p. In the first table the marks are arranged according to ascending powers of i. In the second table the marks are arranged so that the symbols $AB \cdots DE$ represent the positive integers in natural order. By means of these two tables it is possible to perform with ease the operations of addition, subtraction, multiplication and division, within the field.

For an exposition of the Galois field theory, see Dickson's Linear Groups, pages 1–54; Jordan's Traité des Substitutions, pages 14–18, pages 156–161; Serret's Algèbre supérieure. For other references on Galois fields and higher irreducible congruences, see the preface to Dickson's Linear Groups.

Example 1. Simplify $(i^7 + i^{13})(i^2 + 3i + 4)$, i being a primitive root of the $GF[7^2]$.

From the first table for $GF[7^2]$, $i^7 = 6i + 1$, $i^{13} = 3i + 3$, $i^2 = i + 4$.

Therefore $i^7 + i^{13} = 9i + 4 = 2i + 4$ (modulo 7) $= i^6$ (by second table).

Also $i^2 + 3i + 4 = 4i + 8 = 4i + 1$ (modulo 7) $= i^{22}$ (by second table).

Therefore $(i^7 + i^{13})(i^2 + 3i + 4) = i^6 \cdot i^{22} = i^{28} = 5i + 1$ (by first table).

Example 2. Simplify $\dfrac{i^{11} + 3i^{79}}{2i^{26} - 3i^{8} + 2}$, i being a primitive root of the $GF[\mathbf{5}^3]$.

From the first table for $GF[\mathbf{5}^3]$, $i^{11} = 3i^2 + 3i$, $i^{79} = 3i^2 + i + 1$, $i^{26} = 3i + 1$, $-i^8 = i^{70} = 3i^2 + 4i + 4$.

Therefore $i^{11} + 3i^{79} = 3i^2 + 3i + 3(3i^2 + i + 1) = 12i^2 + 6i + 3 = 2i + i + 3$ (mod. 5) $= i^{81}$ (by second table).

Also $2i^{26} - 3i^8 + 2 = 2(3i + 1) + 3(3i^2 + 4i + 4) + 2 = 9i^2 + 18i + 16 = 4i^2 + 3i + 1$ (mod. 5) $= i^{18}$ (by second table).

Therefore

$$\frac{i^{11} + i^{79}}{2i^{26} - 3i^{8} + 2} = \frac{i^{81}}{i^{18}} = i^{63} = 4i \text{ (by first table).}$$

Example 3. Simplify $\dfrac{i^{88} - 10i^7 + 4}{i^{83} + 2/i^{117}}$, i being a primitive root of the $GF[\mathbf{11}^2]$.

From the first table for $GF[\mathbf{11}^2]$, $i^{88} = 6i + 2$, $i^7 = 9i + 2$, $i^{83} = 2i + 3$, $i^3 = 3i + 3$.

Also $-10i^7 = i^7$, because $-10 \equiv 1$ modulo 11; and $2/i^{117} = 2i^3$ because $i^{120} = 1$.

Therefore $i^{88} - 10i^7 + 4 = (6i + 2) + (9i + 2) + 4 = 15i + 8 = 4i + 8$ (modulo 11) $= i^{82}$ (by second table).

Also $i^{83} + 2/i^{117} = (2i + 3) + 2(3i + 3) = 8i + 9 = i^{101}$ (by second table).

Therefore

$$\frac{i^{88} - 10i^7 + 4}{i^{83} + 2/i^{117}} = \frac{i^{82}}{i^{101}} = \frac{i^{82} \cdot i^{120}}{i^{101}} = \frac{i^{202}}{i^{101}} = i^{101} = 8i + 9 \text{ (by first table).}$$

$GF[\mathbf{2}^3]$, $i^3 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ
1		1	0
2	1	0	0
3		1	1
4	1	1	0
5	1	1	1
6	1	0	1
7			1

SECOND TABLE.

λ	α	β	γ
7			1
1		1	0
3		1	1
2	1	0	0
6	1	0	1
4	1	1	0
5	1	1	1

$GF[3^2]$, $i^2 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i + \beta$.

First Table.

λ	α	β	λ	α	β
1	1	0	5	2	0
2	1	1	6	2	2
3	2	1	7	1	2
4		2	8		1

Second Table.

λ	α	β	λ	α	β
8		1	7	1	2
4		2	5	2	0
1	1	0	3	2	1
2	1	1	6	2	2

$GF[2^4]$, $i^4 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

First Table.

λ	α	β	γ	δ
1			1	0
2		1	0	0
3	1	0	0	0
4			1	1
5		1	1	0
6	1	1	0	0
7	1	0	1	1
8		1	0	1
9	1	0	1	0
10		1	1	1
11	1	1	1	0
12	1	1	1	1
13	1	1	0	1
14	1	0	0	1
15				1

Second Table.

λ	α	β	γ	δ
15				1
1			1	0
4			1	1
2		1	0	0
8		1	0	1
5		1	1	0
10		1	1	1
3	1	0	0	0
14	1	0	0	1
9	1	0	1	0
7	1	0	1	1
6	1	1	0	0
13	1	1	0	1
11	1	1	1	0
12	1	1	1	1

$GF[5^2]$, $i^2 \equiv 2i + 2$, modulo 5. $i^\lambda = \alpha i + \beta$.

First Table.

λ	α	β	λ	α	β
1	1	0	13	4	0
2	2	2	14	3	3
3	1	4	15	4	1
4	1	2	16	4	3
5	4	2	17	1	3
6		3	18		2
7	3	0	19	2	0
8	1	1	20	4	4
9	3	2	21	2	3
10	3	1	22	2	4
11	2	1	23	3	4
12		4	24		1

Second Table.

λ	α	β	λ	α	β
24		1	21	2	3
18		2	22	2	4
6		3	7	3	0
12		4	10	3	1
1	1	0	9	3	2
8	1	1	14	3	3
4	1	2	23	3	4
17	1	3	13	4	0
3	1	4	15	4	1
19	2	0	5	4	2
11	2	1	16	4	3
2	2	2	20	4	4

$GF[2^5]$, $i^5 \equiv i^3 + i^2 + i + 1$, modulo 2. $i^\lambda = \alpha i^4 + \beta i^3 + \gamma i^2 + \delta i + \epsilon$.

FIRST TABLE.

λ	α	β	γ	δ	ε
1				1	0
2			1	0	0
3		1	0	0	0
4	1	0	0	0	0
5		1	1	1	1
6	1	1	1	1	0
7	1	0	0	1	1
8		1	0	0	1
9	1	0	0	1	0
10		1	0	1	1
11	1	0	1	1	0
12				1	1
13			1	1	0
14		1	1	0	0
15	1	1	0	0	0
16	1	1	1	1	1
17	1	0	0	0	1
18		1	1	0	1
19	1	1	0	1	0
20	1	1	0	1	1
21	1	1	0	0	1
22	1	1	1	0	1
23	1	0	1	0	1
24			1	0	1
25		1	0	1	0
26	1	0	1	0	0
27			1	1	1
28		1	1	1	0
29	1	1	1	0	0
30	1	0	1	1	1
31					1

SECOND TABLE.

λ	α	β	γ	δ	ε
31					1
1				1	0
12				1	1
2			1	0	0
24			1	0	1
13			1	1	0
27			1	1	1
3		1	0	0	0
8		1	0	0	1
25		1	0	1	0
10		1	0	1	1
14		1	1	0	0
18		1	1	0	1
28		1	1	1	0
5		1	1	1	1
4	1	0	0	0	0
17	1	0	0	0	1
9	1	0	0	1	0
7	1	0	0	1	1
26	1	0	1	0	0
23	1	0	1	0	1
11	1	0	1	1	0
30	1	0	1	1	1
15	1	1	0	0	0
21	1	1	0	0	1
19	1	1	0	1	0
20	1	1	0	1	1
29	1	1	1	0	0
22	1	1	1	0	1
6	1	1	1	1	0
16	1	1	1	1	1

$GF[3^3]$, $i^3 \equiv i + 2$, modulo 3. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
1		1	0	10	1	1	0	19	2	2	2
2	1	0	0	11	1	1	2	20	2	1	1
3		1	2	12	1	0	2	21	1	0	1
4	1	2	0	13			2	22		2	2
5	2	1	2	14		2	0	23	2	2	0
6	1	1	1	15	2	0	0	24	2	2	1
7	1	2	2	16		2	1	25	2	0	1
8	2	0	2	17	2	1	0	26			1
9		1	1	18	1	2	1				

$GF[3^3]$, $i^\lambda = \alpha i^2 + \beta i + \gamma$.

SECOND TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
26			1	21	1	0	1	25	2	0	1
13			2	12	1	0	2	8	2	0	2
1		1	0	10	1	1	0	17	2	1	0
9		1	1	6	1	1	1	20	2	1	1
3		1	2	11	1	1	2	5	2	1	2
14		2	0	4	1	2	0	23	2	2	0
16		2	1	18	1	2	1	24	2	2	1
22		2	2	7	1	2	2	19	2	2	2
2	1	0	0	15	2	0	0				

$GF[7^2]$, $i^2 \equiv i + 4$, modulo 7. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	25	6	0
2	1	4	26	6	3
3	5	4	27	2	3
4	2	6	28	5	1
5	1	1	29	6	6
6	2	4	30	5	3
7	6	1	31	1	6
8		3	32		4
9	3	0	33	4	0
10	3	5	34	4	2
11	1	5	35	6	2
12	6	4	36	1	3
13	3	3	37	4	4
14	6	5	38	1	2
15	4	3	39	3	4
16		2	40		5
17	2	0	41	5	0
18	2	1	42	5	6
19	3	1	43	4	6
20	4	5	44	3	2
21	2	2	45	5	5
22	4	1	46	3	6
23	5	2	47	2	5
24		6	48		1

SECOND TABLE.

λ	α	β	λ	α	β
48		1	39	3	4
16		2	10	3	5
8		3	46	3	6
32		4	33	4	0
40		5	22	4	1
24		6	34	4	2
1	1	0	15	4	3
5	1	1	37	4	4
38	1	2	20	4	5
36	1	3	43	4	6
2	1	4	41	5	0
11	1	5	28	5	1
31	1	6	23	5	2
17	2	0	30	5	3
18	2	1	3	5	4
21	2	2	45	5	5
27	2	3	42	5	6
6	2	4	25	6	0
47	2	5	7	6	1
4	2	6	35	6	2
9	3	0	26	6	3
19	3	1	12	6	4
44	3	2	14	6	5
13	3	3	29	6	6

$GF[2^6]$, $i^6 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \varsigma$.

FIRST TABLE.

λ	α	β	γ	δ	ε	ς	λ	α	β	γ	δ	ε	ζ
1					1	0	33		1	0	0	1	0
2				1	0	0	34	1	0	0	1	0	0
3			1	0	0	0	35			1	0	1	1
4		1	0	0	0	0	36		1	0	1	1	0
5	1	0	0	0	0	0	37	1	0	1	1	0	0
6					1	1	38		1	1	0	1	1
7				1	1	0	39	1	1	0	1	1	0
8			1	1	0	0	40	1	0	1	1	1	1
9		1	1	0	0	0	41		1	1	1	0	1
10	1	1	0	0	0	0	42	1	1	1	0	1	0
11	1	0	0	0	1	1	43	1	1	0	1	1	1
12				1	0	1	44	1	0	1	1	0	1
13			1	0	1	0	45		1	1	0	0	1
14		1	0	1	0	0	46	1	1	0	0	1	0
15	1	0	1	0	0	0	47	1	0	0	1	1	1
16		1	0	0	1	1	48			1	1	0	1
17	1	0	0	1	1	0	49		1	1	0	1	0
18			1	1	1	1	50	1	1	0	1	0	0
19		1	1	1	1	0	51	1	0	1	0	1	1
20	1	1	1	1	0	0	52		1	0	1	0	1
21	1	1	1	0	1	1	53	1	0	1	0	1	0
22	1	1	0	1	0	1	54		1	0	1	1	1
23	1	0	1	0	0	1	55	1	0	1	1	1	0
24		1	0	0	0	1	56		1	1	1	1	1
25	1	0	0	0	1	0	57	1	1	1	1	1	0
26				1	1	1	58	1	1	1	1	1	1
27			1	1	1	0	59	1	1	1	1	0	1
28		1	1	1	0	0	60	1	1	1	0	0	1
29	1	1	1	0	0	0	61	1	1	0	0	0	1
30	1	1	0	0	1	1	62	1	0	0	0	0	1
31	1	0	0	1	0	1	63						1
32			1	0	0	1							

SECOND TABLE.

λ	α	β	γ	δ	ε	ζ	λ	α	β	γ	δ	ε	ζ
63						1	27			1	1	1	0
1					1	0	18			1	1	1	1
6					1	1	4		1	0	0	0	0
2				1	0	0	24		1	0	0	0	1
12				1	0	1	33		1	0	0	1	0
7				1	1	0	16		1	0	0	1	1
26				1	1	1	14		1	0	1	0	0
3			1	0	0	0	52		1	0	1	0	1
32			1	0	0	1	36		1	0	1	1	0
13			1	0	1	0	54		1	0	1	1	1
35			1	0	1	1	9		1	1	0	0	0
8			1	1	0	0	45		1	1	0	0	1
48			1	1	0	1	49		1	1	0	1	0

$GF[2^6]$, $i^6 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	ϵ	ζ	λ	α	β	γ	δ	ϵ	ζ
38		1	1	0	1	1	55	1	0	1	1	1	0
28		1	1	1	0	0	40	1	0	1	1	1	1
41		1	1	1	0	1	10	1	1	0	0	0	0
19		1	1	1	1	0	61	1	1	0	0	0	1
56		1	1	1	1	1	46	1	1	0	0	1	0
5	1	0	0	0	0	0	30	1	1	0	0	1	1
62	1	0	0	0	0	1	50	1	1	0	1	0	0
25	1	0	0	0	1	0	22	1	1	0	1	0	1
11	1	0	0	0	1	1	39	1	1	0	1	1	0
34	1	0	0	1	0	0	43	1	1	0	1	1	1
31	1	0	0	1	0	1	29	1	1	1	0	0	0
17	1	0	0	1	1	0	60	1	1	1	0	0	1
47	1	0	0	1	1	1	42	1	1	1	0	1	0
15	1	0	1	0	0	0	21	1	1	1	0	1	1
23	1	0	1	0	0	1	20	1	1	1	1	0	0
53	1	0	1	0	1	0	59	1	1	1	1	0	1
51	1	0	1	0	1	1	57	1	1	1	1	1	0
37	1	0	1	1	0	0	58	1	1	1	1	1	1
44	1	0	1	1	0	1							

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.

λ	α	β	γ	δ	λ	α	β	γ	δ
1			1	0	24		2	2	2
2		1	0	0	25	2	2	2	0
3	1	0	0	0	26			2	2
4	2	2	1	1	27		2	2	0
5		2	0	2	28	2	2	0	0
6	2	0	2	0	29		1	2	2
7	1	0	2	2	30	1	2	2	0
8	2	1	0	1	31	1	1	1	1
9	2	1	0	2	32			2	1
10	2	1	1	2	33		2	1	0
11	2	2	1	2	34	2	1	0	0
12		2	1	2	35	2	1	2	2
13	2	1	2	0	36	2	0	1	2
14	2	0	2	2	37	1	2	1	2
15	1	0	1	2	38	1	0	0	1
16	2	0	0	1	39	2	2	2	1
17	1	1	0	2	40				2
18		2	0	1	41			2	0
19	2	0	1	0	42		2	0	0
20	1	2	2	2	43	2	0	0	0
21	1	1	0	1	44	1	1	2	2
22		2	2	1	45		1	0	1
23	2	2	1	0	46	1	0	1	0

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	δ	λ	α	β	γ	δ
47	2	0	1	1	64		1	1	1
48	1	2	0	2	65	1	1	1	0
49	1	2	0	1	66			1	1
50	1	2	2	1	67		1	1	0
51	1	1	2	1	68	1	1	0	0
52		1	2	1	69		2	1	1
53	1	2	1	0	70	2	1	1	0
54	1	0	1	1	71	2	2	2	2
55	2	0	2	1	72			1	2
56	1	0	0	2	73		1	2	0
57	2	2	0	1	74	1	2	0	0
58		1	0	2	75	1	2	1	1
59	1	0	2	0	76	1	0	2	1
60	2	1	1	1	77	2	1	2	1
61	2	2	0	2	78	2	0	0	2
62		1	1	2	79	1	1	1	2
63	1	1	2	0	80				1

SECOND TABLE.

λ	α	β	γ	δ	λ	α	β	γ	δ
80				1	3	1	0	0	0
40				2	38	1	0	0	1
1			1	0	56	1	0	0	2
66			1	1	46	1	0	1	0
72			1	2	54	1	0	1	1
41			2	0	15	1	0	1	2
32			2	1	59	1	0	2	0
26			2	2	76	1	0	2	1
2		1	0	0	7	1	0	2	2
45		1	0	1	68	1	1	0	0
58		1	0	2	21	1	1	0	1
67		1	1	0	17	1	1	0	2
64		1	1	1	65	1	1	1	0
62		1	1	2	31	1	1	1	1
73		1	2	0	79	1	1	1	2
52		1	2	1	63	1	1	2	0
29		1	2	2	51	1	1	2	1
42		2	0	0	44	1	1	2	2
18		2	0	1	74	1	2	0	0
5		2	0	2	49	1	2	0	1
33		2	1	0	48	1	2	0	2
69		2	1	1	53	1	2	1	0
12		2	1	2	75	1	2	1	1
27		2	2	0	37	1	2	1	2
22		2	2	1	30	1	2	2	0
24		2	2	2					

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	λ	α	β	γ	δ
50	1	2	2	1	60	2	1	1	1
20	1	2	2	2	10	2	1	1	2
43	2	0	0	0	13	2	1	2	0
16	2	0	0	1	77	2	1	2	1
78	2	0	0	2	35	2	1	2	2
19	2	0	1	0	28	2	2	0	0
47	2	0	1	1	57	2	2	0	1
36	2	0	1	2	61	2	2	0	2
6	2	0	2	0	23	2	2	1	0
55	2	0	2	1	4	2	2	1	1
14	2	0	2	2	11	2	2	1	2
34	2	1	0	0	25	2	2	2	0
8	2	1	0	1	39	2	2	2	1
9	2	1	0	2	71	2	2	2	2
70	2	1	1	0					

$GF[11^2]$, $i^2 \equiv 4i + 9$, modulo 11. $i^\lambda = ai + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
1	1	0	31	3	8	61	10	0	91	8	3
2	4	9	32	9	5	62	7	2	92	2	6
3	3	3	33	8	4	63	8	8	93	3	7
4	4	5	34	3	6	64	7	6	94	8	5
5	10	3	35	7	5	65	1	8	95	4	6
6	10	2	36		8	66	1	9	96		3
7	9	2	37	8	0	67	2	9	97	3	0
8	5	4	38	10	6	68	6	7	98	1	5
9	2	1	39	2	2	69	9	10	99	9	9
10	9	7	40	10	7	70	2	4	100	1	4
11	10	4	41	3	2	71	1	7	101	8	9
12		2	42	3	5	72		9	102	8	6
13	2	0	43	6	5	73	9	0	103	5	6
14	8	7	44	7	10	74	3	4	104	4	1
15	6	6	45	5	8	75	5	5	105	6	3
16	8	10	46	6	1	76	3	1	106	5	10
17	9	6	47	3	10	77	2	5	107	8	1
18	9	4	48		5	78	2	7	108		6
19	7	4	49	5	0	79	4	7	109	6	0
20	10	8	50	9	1	80	1	3	110	2	10
21	4	2	51	4	4	81	7	9	111	7	7
22	7	3	52	9	3	82	4	8	112	2	8
23	9	8	53	6	4	83	2	3	113	5	7
24		4	54	6	10	84		7	114	5	1
25	4	0	55	1	10	85	7	0	115	10	1
26	5	3	56	3	9	86	6	8	116	8	2
27	1	1	57	10	5	87	10	10	117	1	6
28	5	9	58	1	2	88	6	2	118	10	9
29	7	1	59	6	9	89	4	10	119	5	2
30	7	8	60		10	90	4	3	120		1

$GF[11^2]$, $i^2 = 4i + 9$, modulo 11. $i^\lambda = \alpha i + \beta$.

SECOND TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
120		1	67	2	9	103	5	6	91	8	3
12		2	110	2	10	113	5	7	33	8	4
96		3	97	3	0	45	5	8	94	8	5
24		4	76	3	1	28	5	9	102	8	6
48		5	41	3	2	106	5	10	14	8	7
108		6	3	3	3	109	6	0	63	8	8
84		7	74	3	4	46	6	1	101	8	9
36		8	42	3	5	88	6	2	16	8	10
72		9	34	3	6	105	6	3	73	9	0
60		10	93	3	7	53	6	4	50	9	1
1	1	0	31	3	8	43	6	5	7	9	2
27	1	1	56	3	9	15	6	6	52	9	3
58	1	2	47	3	10	68	6	7	18	9	4
80	1	3	25	4	0	86	6	8	32	9	5
100	1	4	104	4	1	59	6	9	17	9	6
98	1	5	21	4	2	54	6	10	10	9	7
117	1	6	90	4	3	85	7	0	23	9	8
71	1	7	51	4	4	29	7	1	99	9	9
65	1	8	4	4	5	62	7	2	69	9	10
66	1	9	95	4	6	22	7	3	61	10	0
55	1	10	79	4	7	19	7	4	115	10	1
13	2	0	82	4	8	35	7	5	6	10	2
9	2	1	2	4	9	64	7	6	5	10	3
39	2	2	89	4	10	111	7	7	11	10	4
83	2	3	49	5	0	30	7	8	57	10	5
70	2	4	114	5	1	81	7	9	38	10	6
77	2	5	119	5	2	44	7	10	40	10	7
92	2	6	26	5	3	37	8	0	20	10	8
78	2	7	8	5	4	107	8	1	118	10	9
112	2	8	75	5	5	116	8	2	87	10	10

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
1		1	0	13	1	0	4	25	2	0	4
2	1	0	0	14		1	3	26		3	1
3		2	3	15	1	3	0	27	3	1	0
4	2	3	0	16	3	2	3	28	1	1	4
5	3	4	1	17	2	4	4	29	1	1	3
6	4	2	4	18	4	3	1	30	1	0	3
7	2	2	2	19	3	4	2	31			3
8	2	1	1	20	4	3	4	32		3	0
9	1	0	1	21	3	2	2	33	3	0	0
10		3	3	22	2	3	4	34		1	4
11	3	3	0	23	3	3	1	35	1	4	0
12	3	1	4	24	3	2	4	36	4	2	3

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
37	2	1	2	67	2	1	4	97	4	1	0
38	1	1	1	68	1	3	1	98	1	3	2
39	1	3	3	69	3	3	3	99	3	4	3
40	3	0	3	70	3	4	4	100	4	4	4
41		4	4	71	4	0	4	101	4	2	2
42	4	4	0	72		2	2	102	2	0	2
43	4	3	2	73	2	2	0	103		1	1
44	3	0	2	74	2	4	1	104	1	1	0
45		3	4	75	4	0	1	105	1	2	3
46	3	4	0	76		4	2	106	2	0	3
47	4	1	4	77	4	2	0	107		2	1
48	1	2	2	78	2	3	2	108	2	1	0
49	2	4	3	79	3	1	1	109	1	4	1
50	4	2	1	80	1	2	4	110	4	3	3
51	2	4	2	81	2	1	3	111	3	1	2
52	4	1	1	82	1	2	1	112	1	3	4
53	1	4	2	83	2	3	3	113	3	1	3
54	4	4	3	84	3	2	1	114	1	4	4
55	4	1	2	85	2	2	4	115	4	1	3
56	1	0	2	86	2	3	1	116	1	1	2
57		4	3	87	3	0	1	117	1	4	3
58	4	3	0	88		2	4	118	4	0	3
59	3	3	2	89	2	4	0	119		1	2
60	3	3	4	90	4	4	1	120	1	2	0
61	3	0	4	91	4	4	2	121	2	2	3
62			4	92	4	0	2	122	2	2	1
63		4	0	93			2	123	2	0	1
64	4	0	0	94		2	0	124			1
65		3	2	95	2	0	0				
66	3	2	0	96		4	1				

SECOND TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
124			1	45		3	4	48	1	2	2
93			2	63		4	0	105	1	2	3
31			3	96		4	1	80	1	2	4
62			4	76		4	2	15	1	3	0
1		1	0	57		4	3	68	1	3	1
103		1	1	41		4	4	98	1	3	2
119		1	2	2	1	0	0	39	1	3	3
14		1	3	9	1	0	1	112	1	3	4
34		1	4	56	1	0	2	35	1	4	0
94		2	0	30	1	0	3	109	1	4	1
107		2	1	13	1	0	4	53	1	4	2
72		2	2	104	1	1	0	117	1	4	3
3		2	3	38	1	1	1	114	1	4	4
88		2	4	116	1	1	2	95	2	0	0
32		3	0	29	1	1	3	123	2	0	1
26		3	1	28	1	1	4	102	2	0	2
65		3	2	120	1	2	0	106	2	0	3
10		3	3	82	1	2	1	25	2	0	4

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
108	2	1	0	61	3	0	4	118	4	0	3
8	2	1	1	27	3	1	0	71	4	0	4
37	2	1	2	79	3	1	1	97	4	1	0
81	2	1	3	111	3	1	2	52	4	1	1
67	2	1	4	113	3	1	3	55	4	1	2
73	2	2	0	12	3	1	4	115	4	1	3
122	2	2	1	66	3	2	0	47	4	1	4
7	2	2	2	84	3	2	1	77	4	2	0
121	2	2	3	21	3	2	2	50	4	2	1
85	2	2	4	16	3	2	3	101	4	2	2
4	2	3	0	24	3	2	4	36	4	2	3
86	2	3	1	11	3	3	0	6	4	2	4
78	2	3	2	23	3	3	1	58	4	3	0
83	2	3	3	59	3	3	2	18	4	3	1
22	2	3	4	69	3	3	3	43	4	3	2
89	2	4	0	60	3	3	4	110	4	3	3
74	2	4	1	46	3	4	0	20	4	3	4
51	2	4	2	5	3	4	1	42	4	4	0
49	2	4	3	19	3	4	2	90	4	4	1
17	2	4	4	99	3	4	3	91	4	4	2
33	3	0	0	70	3	4	4	54	4	4	3
87	3	0	1	64	4	0	0	100	4	4	4
44	3	0	2	75	4	0	1				
40	3	0	3	92	4	0	2				

$GF[2^7]$, $i^7 \equiv i + 1$, modulo 2. $i^{-\lambda} = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

FIRST TABLE.

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
1						1	0	24	1	1	1	1	0	0	0
2					1	0	0	25	1	1	1	0	0	1	1
3				1	0	0	0	26	1	1	0	0	1	0	1
4			1	0	0	0	0	27	1	0	0	1	0	0	1
5		1	0	0	0	0	0	28			1	0	0	0	1
6	1	0	0	0	0	0	0	29		1	0	0	0	1	0
7						1	1	30	1	0	0	0	1	0	0
8					1	1	0	31				1	0	1	1
9				1	1	0	0	32			1	0	1	1	0
10			1	1	0	0	0	33		1	0	1	1	0	0
11		1	1	0	0	0	0	34	1	0	1	1	0	0	0
12	1	1	0	0	0	0	0	35		1	1	0	0	1	1
13	1	0	0	0	0	1	1	36	1	1	0	0	1	1	0
14					1	0	1	37	1	0	0	1	1	1	1
15				1	0	1	0	38			1	1	1	0	1
16			1	0	1	0	0	39		1	1	1	0	1	0
17		1	0	1	0	0	0	40	1	1	1	0	1	0	0
18	1	0	1	0	0	0	0	41	1	1	0	1	0	1	1
19		1	0	0	0	1	1	42	1	0	1	0	1	0	1
20	1	0	0	0	1	1	0	43		1	0	1	0	0	1
21				1	1	1	1	44	1	0	1	0	0	1	0
22			1	1	1	1	0	45		1	0	0	1	1	1
23		1	1	1	1	0	0	46	1	0	0	1	1	1	0

$GF[2^7]$, $i^7 \equiv i + 1$, modulo 2. $i^{-\lambda} = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \varepsilon i^2 + \zeta i + \eta$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
47			1	1	1	1	1	88	1	1	0	0	0	1	0
48		1	1	1	1	1	0	89	1	0	0	0	1	1	1
49	1	1	1	1	1	0	0	90				1	1	0	1
50	1	1	1	1	0	1	1	91			1	1	0	1	0
51	1	1	1	0	1	0	1	92		1	1	0	1	0	0
52	1	1	0	1	0	0	1	93	1	1	0	1	0	0	0
53	1	0	1	0	0	0	1	94	1	0	1	0	0	1	1
54		1	0	0	0	0	1	95		1	0	0	1	0	1
55	1	0	0	0	0	1	0	96	1	0	0	1	0	1	0
56					1	1	1	97			1	0	1	1	1
57				1	1	1	0	98		1	0	1	1	1	0
58			1	1	1	0	0	99	1	0	1	1	1	0	0
59		1	1	1	0	0	0	100		1	1	1	0	1	1
60	1	1	1	0	0	0	0	101	1	1	1	0	1	1	0
61	1	1	0	0	0	1	1	102	1	1	0	1	1	1	1
62	1	0	0	0	1	0	1	103	1	0	1	1	1	0	1
63				1	0	0	1	104		1	1	1	0	0	1
64			1	0	0	1	0	105	1	1	1	0	0	1	0
65		1	0	0	1	0	0	106	1	1	0	0	1	1	1
66	1	0	0	1	0	0	0	107	1	0	0	1	1	0	1
67			1	0	0	1	1	108			1	1	0	0	1
68		1	0	0	1	1	0	109		1	1	0	0	1	0
69	1	0	0	1	1	0	0	110	1	1	0	0	1	0	0
70			1	1	0	1	1	111	1	0	0	1	0	1	1
71		1	1	0	1	1	0	112			1	0	1	0	1
72	1	1	0	1	1	0	0	113		1	0	1	0	1	0
73	1	0	1	1	0	1	1	114	1	0	1	0	1	0	0
74		1	1	0	1	0	1	115		1	0	1	0	1	1
75	1	1	0	1	0	1	0	116	1	0	1	0	1	1	0
76	1	0	1	0	1	1	1	117		1	0	1	1	1	1
77		1	0	1	1	0	1	118	1	0	1	1	1	1	0
78	1	0	1	1	0	1	0	119		1	1	1	1	1	1
79		1	1	0	1	1	1	120	1	1	1	1	1	1	0
80	1	1	0	1	1	1	0	121	1	1	1	1	1	1	1
81	1	0	1	1	1	1	1	122	1	1	1	1	1	0	1
82		1	1	1	1	0	1	123	1	1	1	1	0	0	1
83	1	1	1	1	0	1	0	124	1	1	1	0	0	0	1
84	1	1	1	0	1	1	1	125	1	1	0	0	0	0	1
85	1	1	0	1	1	0	1	126	1	0	0	0	0	0	1
86	1	0	1	1	0	0	1	127							1
87		1	1	0	0	0	1								

$GF[2^7]$, $i^7 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

SECOND TABLE.

λ	α	β	γ	δ	ϵ	ζ	η	λ	α	β	γ	δ	ϵ	ζ	η
127							1	87		1	1	0	0	0	1
1						1	0	109		1	1	0	0	1	0
7						1	1	35		1	1	0	0	1	1
2					1	0	0	92		1	1	0	1	0	0
14					1	0	1	74		1	1	0	1	0	1
8					1	1	0	71		1	1	0	1	1	0
56					1	1	1	79		1	1	0	1	1	1
3				1	0	0	0	59		1	1	1	0	0	0
63				1	0	0	1	104		1	1	1	0	0	1
15				1	0	1	0	39		1	1	1	0	1	0
31				1	0	1	1	100		1	1	1	0	1	1
9				1	1	0	0	23		1	1	1	1	0	0
90				1	1	0	1	82		1	1	1	1	0	1
57				1	1	1	0	48		1	1	1	1	1	0
21				1	1	1	1	119		1	1	1	1	1	1
4			1	0	0	0	0	6	1	0	0	0	0	0	0
28			1	0	0	0	1	126	1	0	0	0	0	0	1
64			1	0	0	1	0	55	1	0	0	0	0	1	0
67			1	0	0	1	1	13	1	0	0	0	0	1	1
16			1	0	1	0	0	30	1	0	0	0	1	0	0
112			1	0	1	0	1	62	1	0	0	0	1	0	1
32			1	0	1	1	0	20	1	0	0	0	1	1	0
97			1	0	1	1	1	89	1	0	0	0	1	1	1
10			1	1	0	0	0	66	1	0	0	1	0	0	0
108			1	1	0	0	1	27	1	0	0	1	0	0	1
91			1	1	0	1	0	96	1	0	0	1	0	1	0
70			1	1	0	1	1	111	1	0	0	1	0	1	1
58			1	1	1	0	0	69	1	0	0	1	1	0	0
38			1	1	1	0	1	107	1	0	0	1	1	0	1
22			1	1	1	1	0	46	1	0	0	1	1	1	0
47			1	1	1	1	1	37	1	0	0	1	1	1	1
5		1	0	0	0	0	0	18	1	0	1	0	0	0	0
54		1	0	0	0	0	1	53	1	0	1	0	0	0	1
29		1	0	0	0	1	0	44	1	0	1	0	0	1	0
19		1	0	0	0	1	1	94	1	0	1	0	0	1	1
65		1	0	0	1	0	0	114	1	0	1	0	1	0	0
95		1	0	0	1	0	1	42	1	0	1	0	1	0	1
68		1	0	0	1	1	0	116	1	0	1	0	1	1	0
45		1	0	0	1	1	1	76	1	0	1	0	1	1	1
17		1	0	1	0	0	0	34	1	0	1	1	0	0	0
43		1	0	1	0	0	1	86	1	0	1	1	0	0	1
113		1	0	1	0	1	0	78	1	0	1	1	0	1	0
115		1	0	1	0	1	1	73	1	0	1	1	0	1	1
33		1	0	1	1	0	0	99	1	0	1	1	1	0	0
77		1	0	1	1	0	1	103	1	0	1	1	1	0	1
98		1	0	1	1	1	0	118	1	0	1	1	1	1	0
117		1	0	1	1	1	1	81	1	0	1	1	1	1	1
11		1	1	0	0	0	0	12	1	1	0	0	0	0	0

$GF[2^7]$, $i^7 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \varepsilon i^2 + \zeta i + \eta$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
125	1	1	0	0	0	0	1	124	1	1	1	0	0	0	1
88	1	1	0	0	0	1	0	105	1	1	1	0	0	1	0
61	1	1	0	0	0	1	1	25	1	1	1	0	0	1	1
110	1	1	0	0	1	0	0	40	1	1	1	0	1	0	0
26	1	1	0	0	1	0	1	51	1	1	1	0	1	0	1
36	1	1	0	0	1	1	0	101	1	1	1	0	1	1	0
106	1	1	0	0	1	1	1	84	1	1	1	0	1	1	1
93	1	1	0	1	0	0	0	24	1	1	1	1	0	0	0
52	1	1	0	1	0	0	1	123	1	1	1	1	0	0	1
75	1	1	0	1	0	1	0	83	1	1	1	1	0	1	0
41	1	1	0	1	0	1	1	50	1	1	1	1	0	1	1
72	1	1	0	1	1	0	0	49	1	1	1	1	1	0	0
85	1	1	0	1	1	0	1	122	1	1	1	1	1	0	1
80	1	1	0	1	1	1	0	120	1	1	1	1	1	1	0
102	1	1	0	1	1	1	1	121	1	1	1	1	1	1	1
60	1	1	1	0	0	0	0								

$GF[13^2]$, $i^2 \equiv i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
1	1	0	25	7	5	49	4	11	73	7	1
2	1	11	26	12	12	50	2	5	74	8	12
3	12	11	27	11	2	51	7	9	75	7	10
4	10	2	28		4	52	3	12	76	4	12
5	12	6	29	4	0	53	2	7	77	3	5
6	5	2	30	4	5	54	9	9	78	8	7
7	7	3	31	9	5	55	5	8	79	2	10
8	10	12	32	1	8	56		3	80	12	9
9	9	6	33	9	11	57	3	0	81	8	2
10	2	8	34	7	8	58	3	7	82	10	10
11	10	9	35	2	12	59	10	7	83	7	6
12	6	6	36	1	9	60	4	6	84		12
13	12	1	37	10	11	61	10	5	85	12	0
14		2	38	8	6	62	2	6	86	12	2
15	2	0	39	1	10	63	8	9	87	1	2
16	2	9	40	11	11	64	4	10	88	3	11
17	11	9	41	9	4	65	1	5	89	1	7
18	7	4	42		8	66	6	11	90	8	11
19	11	12	43	8	0	67	4	1	91	6	10
20	10	4	44	8	10	68	5	5	92	3	1
21	1	6	45	5	10	69	10	3	93	4	7
22	7	11	46	2	3	70		6	94	11	5
23	5	12	47	5	9	71	6	0	95	3	4
24	4	3	48	1	3	72	6	1	96	7	7

$GF[13^2]$, $i^2 \equiv i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.—*Continued.*

λ	α	β	λ	α	β	λ	α	β	λ	α	β
97	1	12	115	4	8	133	9	2	151	9	12
98		11	116	12	5	134	11	8	152	8	8
99	11	0	117	4	2	135	6	4	153	3	10
100	11	4	118	6	5	136	10	1	154		7
101	2	4	119	11	1	137	11	6	155	7	0
102	6	9	120	12	4	138	4	4	156	7	12
103	2	1	121	3	2	139	8	5	157	6	12
104	3	9	122	5	7	140		10	158	5	1
105	12	7	123	12	3	141	10	0	159	6	3
106	6	2	124	2	2	142	10	6	160	9	1
107	8	1	125	4	9	143	3	6	161	10	8
108	9	10	126		5	144	9	7	162	5	6
109	6	8	127	5	0	145	3	8	163	11	3
110	1	1	128	5	3	146	11	7	164	1	4
111	2	11	129	8	3	147	5	4	165	5	11
112		9	130	11	10	148	9	3	166	3	3
113	9	0	131	8	4	149	12	8	167	6	7
114	9	8	132	12	10	150	7	2	168		1

SECOND TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
168		1	15	2	0	52	3	12	165	5	11
14		2	103	2	1	29	4	0	23	5	12
56		3	124	2	2	67	4	1	71	6	0
28		4	46	2	3	117	4	2	72	6	1
126		5	101	2	4	24	4	3	106	6	2
70		6	50	2	5	138	4	4	159	6	3
154		7	62	2	6	30	4	5	135	6	4
42		8	53	2	7	60	4	6	118	6	5
112		9	10	2	8	93	4	7	12	6	6
140		10	16	2	9	115	4	8	167	6	7
98		11	79	2	10	125	4	9	109	6	8
84		12	111	2	11	64	4	10	102	6	9
1	1	0	35	2	12	49	4	11	91	6	10
110	1	1	57	3	0	76	4	12	66	6	11
87	1	2	92	3	1	127	5	0	157	6	12
48	1	3	121	3	2	158	5	1	155	7	0
164	1	4	166	3	3	6	5	2	73	7	1
65	1	5	95	3	4	128	5	3	150	7	2
21	1	6	77	3	5	147	5	4	7	7	3
89	1	7	143	3	6	68	5	5	18	7	4
32	1	8	58	3	7	162	5	6	25	7	5
36	1	9	145	3	8	122	5	7	83	7	6
39	1	10	104	3	9	55	5	8	96	7	7
2	1	11	153	3	10	47	5	9	34	7	8
97	1	12	88	3	11	45	5	10	51	7	9

$GF[13^2]$, $i^2 \equiv i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

SECOND TABLE.—*Continued.*

λ	α	β	λ	α	β	λ	α	β	λ	α	β
75	7	10	160	9	1	61	10	5	17	11	9
22	7	11	133	9	2	142	10	6	130	11	10
156	7	12	148	9	3	59	10	7	40	11	11
43	8	0	41	9	4	161	10	8	19	11	12
107	8	1	31	9	5	11	10	9	85	12	0
81	8	2	9	9	6	82	10	10	13	12	1
129	8	3	144	9	7	37	10	11	86	12	2
131	8	4	114	9	8	8	10	12	123	12	3
139	8	5	54	9	9	99	11	0	120	12	4
38	8	6	108	9	10	119	11	1	116	12	5
78	8	7	33	9	11	27	11	2	5	12	6
152	8	8	151	9	12	163	11	3	105	12	7
63	8	9	141	10	0	100	11	4	149	12	8
44	8	10	136	10	1	94	11	5	80	12	9
90	8	11	4	10	2	137	11	6	132	12	10
74	8	12	69	10	3	146	11	7	3	12	11
113	9	0	20	10	4	134	11	8	26	12	12

NOTES.

THE July number (volume 6, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Sur les lignes géodésiques des surfaces convexes," by H. POINCARÉ; "The classification of quadrics," by T. J. I'A. BROMWICH; "On differential invariants," by J. E. WRIGHT; "Groups of order p^m, which contain cyclic subgroups of order p^{m-3}," by L. I. NEIKIRK; "On the invariant subgroups of prime index," by G. A. MILLER; "On a general method for treating transmitted motions and its application to indirect perturbations," by E. W. BROWN; "On hypercomplex number systems," by L. E. DICKSON; "A theorem on finite algebras," by J. H. MACLAGAN-WEDDERBURN; "The relation of the principles of logic to the foundations of geometry," by J. ROYCE; "On multiple integrals," by J. PIERPONT.

THE July number (volume 27, number 3) of the *American Journal of Mathematics* contains: "Deduction of the power series representing a function from special values of the latter," by G. W. HILL; "On the definition of reducible hypercomplex number systems," by S. EPSTEEN and H. B. LEONARD;