between the coördinates $\bar{x}$, $\bar{y}$ in the plane of projection, and the minimal parameters $\bar{u}$, $\bar{v}$ on the sphere. These relations yield, as the equation of the curves into which the loxodromes project,

$$\cos^{-1}\frac{\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2}} = A \log \sqrt{\bar{x}^2+\bar{y}^2} + B$$

or, in polar coördinates,

$$\rho = \alpha e^{\beta\theta} \qquad (\alpha, \beta = \text{const.}).$$

The case of the oblate spheroid is interesting, in that the loxodromes on it project stereographically into the same spirals as do the loxodromes on the sphere which is tangent to the spheroid along the equator.

BERKELEY, *October*, 1905.

STOLZ AND GMEINER'S FUNCTION THEORY.

Einleitung in die Funktionentheorie. Abteilung I. By OTTO STOLZ and J. ANTON GMEINER. Leipzig, B. G. Teubner. 1904. vi + 242 pp.

THE Funktionentheorie of Stolz and Gmeiner is a working over of certain parts of Stolz's Allgemeine Arithmetik which do not appear in the new Stolz and Gmeiner's Theoretische Arithmetik. The two new books are evidently to be thought of together as a single course in the elements of analysis. The contents of the Theoretische Arithmetik correspond in a general way to sections 1 to 7, part of 10, and most of 11 of volume I, and of sections 1, 2, 6, and part of 5 of volume II of the Allgemeine Arithmetik. The Theoretische Arithmetik begins with the theory of whole numbers ; then, after discussing the system of rational numbers, positive and negative, develops for the real and ordinary complex numbers the theory of addition, subtraction, multiplication, division, exponents and logarithms. In the course of this development appears a short discussion of complex numbers of n units, including quaternions, some geometric applications, and the fundamental theorems on infinite series.

The Funktionentheorie, on the other hand, corresponds to sections 9, 11, and part of 10 of volume I, and to sections 3, 4, 7, 8, and part of 5 of volume II of the Allgemeine Arithmetik. The first Abteilung, which is before us, contains the general subjects of functions of one and of several variables, and the special subjects of rational integral functions and of power series in one or two variables. In the second part the authors promise to treat criteria for the convergence and divergence of series, Weierstrass's theory of analytic functions of one variable, circular functions, infinite products, finite and infinite continued fractions.

Function theory is understood in a different sense from that of mere continuity considerations. For example, Abschnitt IV, on rational integral functions, contains an extensive portion of what is ordinarily called algebra — divisibility, highest common factor, etc. On the other hand, the operations of infinitesimal calculus are excluded. $D_x f(x)$ is indeed defined for rational integral functions and for power series, but by an algebraic definition, and not by a direct limiting process.

The intention of the authors seems not to be to develop the exceedingly general theorems that are possible in function theory and then to get the applications to particular functions from these as special cases. They rather aim to complete the theory of the simpler functions alone, proving the general theorems only where they are needed for the cases of rational functions or of analytic functions. The exposition of real function theory in the ordinary sense goes but little beyond the definitions.

The argument in favor of this mode of exposition is of course that in order to understand the "ordinary functions" one does not need or wish to give his attention to the refinements necessary for the general theorems. But there are certain general theorems, treated in a very cursory way by our authors, to which this argument does not seem to apply with great force. For example, in proving Weierstrass's theorem on double series (page 206) they find it necessary to go clear back to first principles instead of referring to a theorem about the equality of $L_x L_y f(x, y)$ and $L_y L_x f(x, y)$ as they might easily have done if their analysis of this equality had been at all profound. The effect in this case is that the reader is subjected to practically all the difficulties of the general theorem in proving the special case at a time when his interest is in the elegant and rapid development of the theory of analytic functions rather than in the niceties of real function theory. Another consideration is that the demonstration of a theorem like the one about

$L_x L_y f(x, y) = L_y L_x f(x, y)$ in a special case is less likely than in the general case to bring out the issue essentially involved and is more likely to impress the reader as a mere cloud of symbols.

Another instance in which it seems to the reviewer that there would be a gain in using general principles of broader scope is the class of theorems expressing connections between the behavior of a function on an interval and at a point or points. Examples are the theorem of uniform continuity and the theorem that if a function $f(x)$ has on an interval a least upper bound b, there will be a point in every neighborhood of which $f(x)$ has the least upper bound b. The method of proof of Stolz and Gmeiner is the familiar one, to divide the interval on which a function is supposed to possess a certain property into e ($e \geqq 2$) equal parts. On at least one of these parts the function has the same property. The process is then repeated, and so on.

To go through this soporific ritual whenever we meet one of these theorems seems to the reviewer just as much in the spirit of modern mathematics as it would be actually to multiply out $(a + b)^n$ whenever we meet it in a special case, instead of using the binomial theorem. The modern way is to prove once for all a general theorem about classes of segments which may be cited in the special cases. One form of such a proposition is the Heine-Borel theorem to which we have referred in a former number of the BULLETIN.* Another useful form is the theorem of Cantor which is to be found on page 48, volume II of the Lezioni di analisi infinitesimale of G. Peano. Of course it will be objected that these theorems about classes of segments, like the theorems about point sets, are very abstract. But one must always have recourse to abstractions if he wishes to avoid tiresome repetitions of the same idea.

Another instance of the tendency of our authors to avoid the more general concepts is that the definition of continuity at a point (page 36) requires that there shall be an interval including the point upon every point of which the function is defined. So far as the reviewer observed there was no theorem whose demonstration was facilitated by this restriction. Another curiosity is in the definition of a variable (page 1): "Unter einer reellen Veränderlichen versteht man ein Zeichen, das unbegrenzt viele reelle Zahlen *deren jede völlig bestimmt sein muss*, bedeuten kann." The phrase which we have italicized

* The Heine-Borel Theorem, BULLETIN, vol. 10, p. 436.

appears again on page 114 in the definition of a complex variable and in other places. But what sort of a thing is a number which is not fully determined? We cannot believe that any clearness is gained by introducing such redundancies.

In "geometrically representing" a real function (pages 61–62) the authors believe it necessary to limit attention to such graphs as are composed of a finite number of straight or of convex curved segments. This, because only such curves can be constructed by mechanical means.

Throughout the book we get the impression that geometry is thought of, not as a science with the same logical ideals as arithmetic, but as a more or less unorganized mass of intuitions. For the purposes of analysis these intuitions are connected mainly with the notion of marking on a blackboard or a piece of paper. A corollary of this comment is that questions of topology (or analysis situs) are avoided as far as possible. Where it is not possible to avoid them, as in the complex variable theory, geometric notions are used without anything like the keen analysis that is expended on the corresponding arithmetical ideas. Compare pages 115, 116, 199.

The attitude toward geometry here complained of, and to a large extent the other objects of complaint, are not characteristic of this book alone, but of the whole extreme arithmetizing tendency in analysis. The criticism is made from the point of view of those who believe that rigor is not much more difficult in geometry than in arithmetic and that by the use of the generalizations and the illuminating language of geometry a much greater elegance can be obtained than by a strictly arithmetical method. It is made moreover with a thorough realization that the points attacked can all be strongly defended.

Commendation of this particular book is the easier on account of the elegance of its literary style. It is well arranged both in its large and in its small divisions. The language is generally clear and simple — though in some of the ϵ and δ arguments the interdependences of the various epsilons are not sufficiently indicated. The references to the sources and history of the theories treated are satisfactorily complete without being too verbose. The definitions are followed by good illustrative examples. Each of the five Abschnitte is followed by an interesting collection of Übungen. There is in general such an effect of dignity and maturity of thought that one feels that the book could be but little improved without changing its whole method.

An interesting question very well handled is the use of the symbols ∞, $+\infty$, $-\infty$. With the real number system can be associated *two* extra symbols, $+\infty$ and $-\infty$, having order relations with the rest and serving as the upper and lower bounds of the number scale (pages 1, 3). With the number system can be associated *one* extra symbol, ∞, having operational relations with the rest (page 6). That is, if the variable x be represented by pairs of numbers $\{x_1, x_2\}$ such that x is represented by the class $[\{x_1, x_2\}]$ for which $x_1/x_2 = x$, then one excludes from consideration $\{0, 0\}$ and represents the class $\{x, 0\}$ by ∞. The symbol ∞ is thus essentially without algebraic sign. On page 10 we find ∞ as a value of a function in case of a finite number of infinite discontinuities and on page 37 we find the notion of continuity at ∞.

The book is well printed, as the books of this publisher generally are. We have noted only three typographical errors. They should be corrected as follows:

Page 78, 18th line from bottom read $\phi(y)$ for $\phi(x)$.

Page 88, 11th line from bottom read $a + \delta_1$ for $a < \delta_1$.

Page 142, 14th line from bottom read $(n - r - 1)$ for $(n - p - r - 1)$.

OSWALD VEBLEN.

PRINCETON, N. J.,
September, 1905.

CESÀRO-KOWALEWSKI'S ALGEBRAIC ANALYSIS AND INFINITESIMAL CALCULUS.

Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung. Von E. CESÀRO. Deutsch von G. KOWALEWSKI. Leipzig, B. G. Teubner, 1904. 8vo. 6 + 894 pp.

THE above work, which was translated from the author's manuscript, is a revision of the Analisi Algebrica (1894) and the Calcolo Infinitesimale (1897). While the text is somewhat changed, the revision consists mostly in rearrangement partly made necessary by publishing the two books as one. The book is the outgrowth of lectures by Cesàro on algebraic analysis given simultaneously with a course on analytic geometry. The student is referred to such writers as Dini, Weber, Stolz, Jordan for more extended discussions of the principles, but the author has given great weight to the application of the principles de-