Areas defined by polar coördinates are first treated by means of transformation of the variables after the rectangular coördinates have been discussed; but a good geometric discussion is added. Maclaurin's series and allied formulas are treated at some length, the forms of the remainder being minutely compared with the forms derived in differential calculus, but the difficulties of a double or multiple limit receive but little attention.

Chapter 2 is concerned with quadrature, complanation, and rectification. The idea of length is carefully defined and a very large number of illustrations are brought in, but all are worked out in full, leaving no problems for the student. The two following chapters are devoted to tangents and normals, contact and curvature, and inflexions, treated by the usual methods of the differential calculus. The author explains in the preface that these subjects were excluded from the preceding volume on account of its size.

It seems remarkable that when nearly three hundred pages have been covered, over half of them dealing with applications, only a very limited number of forms of integrals have been derived. Much space could have been spared by a different arrangement. In the latter part of the book (pages 281–370) a long chapter is devoted to systematic integration, very similar to the plan followed by the better American text-books, though many more details are given, leaving fewer steps for the student to supply. As a natural extension of this chapter another follows on elliptic integrals, the addition theorems of elliptic, hyperbolic, and circular functions, and integration in series. The last topic, double integration, occupies about twenty pages. These include the fundamental definitions, discussion of limits, change of sequence, transformations of the variables, and a few applications to cubature and complanation.

The book contains a number of instructive features, many problems are treated by new methods, but the arrangement of topics seems unfortunate for a student. The book is full of helpful cross references, but is not provided with an index.

VIRGIL SNYDER.

*Le Calcul des Résidus et ses Applications à la Théorie des Fonctions.* By ERNST LINDELÖF. Paris, Gauthier-Villars, 1905. 141 pp.

WE here have a small volume devoted to a relatively narrow though important field of mathematics. The calculus of residues, as used in the sense of the author, comprises such results as flow in a direct manner from Cauchy's integral theorem for functions of a complex variable — *i. e.* from the theorem "the integral of a function $f(z)$ of complex variable $z$ about a closed contour upon which this function is single-valued and analytic and within which the same function is single-valued and analytic except for the poles $z = \lambda_1, \lambda_2, \cdots, \lambda_n$ is equal to the sum of the residues of these poles." A familiar example of such results is furnished by the application so frequently made of the theorem for the evaluation of definite integrals, an application due to Cauchy himself and highly developed by him. Likewise, it has long been known that the theorem may be made to yield valuable results in the study of the convergence of Fourier's series. Within recent years Cauchy's investigations in this field have been worked over and many additions have been made to the list of applications of the theorem, which list now includes convergence proofs for several of the fundamental series developments of mathematical physics and important additions to our knowledge of the properties of functions defined by power series. As the field has widened and its importance has become more and more recognized, the need of some systematic exposition of it has naturally been created and we may therefore at once extend to this pioneer volume of Professor Lindelöf a hearty welcome.

Chapter I (Principes et théorèmes fondamentaux, pages 1–20) contains such material from the general theory of functions as is needed for reference in the subsequent pages. It seems somewhat surprising, in view of the special character of the book, that the author should take the trouble here to give proofs. For example, it appears superfluous in such a work to *prove* Laurent's theorem. A mere tabulation of the needed fundamental theorems and definitions would apparently suffice.

In Chapter II (Applications diverses du calcul des résidus, pages 20–52) we soon see how Cauchy's integral theorem yields a wide variety of interesting relations, some old and some new. For example, it thus appears that for the $k$th Bernoulli number $B_k$ we may write

$$B_k = \frac{2(2k)!}{(2\pi)^{2k}} \sum_1^\infty \frac{1}{\nu^{2k}}.$$

The chapter closes with nine pages devoted to the study of definite integrals (after the celebrated method of Cauchy) from the standpoint of the calculus of residues. By means of well-chosen examples it is seen how powerful the method is, and the reader is thus left with a clear impression of the assistance derived from the theory of functions of a complex variable in the study of functions of a real variable.

In Chapter III (Formules sommatoires tirées du calcul des résidus, pages 52–87) the main object of discussion is the infinite series

$$\sum_m^\infty f(\nu),$$

$f(\nu)$ being a function of $\nu$ subject to certain preliminary conditions. When these conditions are satisfied it is shown that the calculus of residues enables one to express the sum of the series by means of a finite number of definite (improper) integrals. For example, we may thus write

$$\sum_m^\infty f(\nu) = \int_a^\infty f(\tau)d\tau + \int_a^{a+i\infty} \frac{f(z)dz}{e^{-2\pi iz}-1} + \int_a^{a-i\infty} \frac{f(z)dz}{e^{2\pi iz}-1}$$
$$(m-1 < a < m).$$

The utility of this formula is at once apparent, for if we can sum a series in this form we are thereby in possession of something from which the properties of the series (in general difficult to obtain) may be deduced. It is in precisely this way that the study of infinite series has been notably enriched within recent years through the calculus of residues. The author, having once established these "formules sommatoires," virtually devotes the remainder of his volume to pointing out their important applications. First, a variety of results due to Euler, Maclaurin, Hermite, Darboux and others are obtained as special consequences, then the bearing of the same formulas upon the study of the convergence of Fourier's series (a bearing well known to Cauchy himself) is briefly pointed out.

Two particular points are perhaps worthy of note before dismissing the chapter. On page 57 where several conditions are imposed upon $f(z)$ the author requires in the first of the conditions numbered 2° that the equality

$$\lim_{t=\pm\infty} e^{-2\pi|t|}f(\tau+it) = 0$$

shall hold true *uniformly* for $a \leqq \tau \leqq \beta$. We would note that the equality signs are here superfluous — *i. e.*, the condition upon $\tau$ may be written simply $\alpha < \tau < \beta$. (Cf. Stolz, Allgemeine Arithmetik, volume I (1885), page 273.) Likewise, the second of the conditions numbered 2° would be better if it read simply "La condition (A) est vérifiée uniformément pour $\tau > a$." Similar remarks may be made at numerous places throughout the work where the notion of uniform convergence is introduced.

Secondly, a fruitful source of investigation is suggested on page 64, where the author calls attention to the alteration required in the formulas just preceding when it is assumed that $f(z)$ possesses a finite number of singular points in the region specified at the top of the page. Instead of the number of such points being finite, suppose a case where they are infinite. Then the required alteration introduces an infinite series of residues into the formula in question. Can we still express

$$\sum_{m}^{\infty} f(\nu)$$

by means of a finite number of definite integrals; if so, when?

Chapter IV (Les fonctions $\Gamma(x)$, $\zeta(s)$, $\zeta(s, w)$, pages 87–108) is devoted principally to the discussion by means of the formulas of chapter III of the function $\log \Gamma(x)$ and its derivatives and of the socalled function of Riemann $\zeta(s)$.

In Chapter V (Applications au prolongement analytique et à l'étude asymptotique des fonctions définies par un développement de Taylor, pages 108–141) we see clearly the bearing of the calculus of residues upon the problem of analytic extension for functions defined by power series: *i. e.*, the problem of determining the value of the function $F(x)$ at any point in its domain of existence when this function is defined by the series

$$F(x) = \phi(0) + \phi(1)x + \phi(2)x^2 + \cdots + \phi(\nu)x^\nu + \cdots$$

(*radius of convergence finite and different from zero*).

Under specified conditions for $\phi(\nu)$ it is found that the function $F(x)$ exists and is analytic at all points $x$ except those which lie within a certain segment of the plane containing the real axis of $x$, and an explicit form for this function in terms of definite integrals is obtained. The chapter for the most part is an exposition of the author's own investigations upon the subject, as contained in his memoir "Quelques applications d'une formule sommatoire générale" (*Acta Societatis Scientiarum Fennicae*, volume 31, number 3 (1902)). It must be admitted that this chapter is difficult reading, a feature which seems due in large measure to the involved character of the hypotheses placed upon $\phi(\nu)$. In this connection may we not question which course would here be preferable, to employ involved hypotheses like these and thus attain a degree of generality in the results as great as that obtained by all previous investigators, or to sacrifice generality in some measure for the sake of simplicity and attractiveness? In a treatise like this the reviewer believes the latter alternative preferable.

Finally, we will venture to make one general remark. Professor Lindelöf's work being confined merely to the applications of the calculus of residues to the theory of functions as such, it is not surprising that nowhere do we find mention of the profound application of this calculus which Dini has made in the study of the convergence of the important series developments of mathematical physics. As early as 1880 his work "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale" appeared, containing rigorous convergence proofs based upon the calculus of residues for Fourier's series, series in terms of Bessel's functions, zonal harmonics and elliptic functions. Attention is especially directed to the portion of this work from page 139 to page 328. Any general treatise bearing the title used by our author should certainly dwell more at length upon this latter aspect of the subject. Let us therefore hope to see in the near future a similar treatise having a wider scope.

WALTER B. FORD.

*Geschichte der Elementar-Mathematik in systematischer Darstellung.* By Dr. JOHANNES TROPFKE, Oberlehrer am Friedrich-Realgymnasium zu Berlin. Bd. I. Rechnen und Algebra, Leipzig, 1902. viii + 332 pp. Bd. II. Geometrie, Logarithmen, Sphärik und Sphärische Trigonometrie, Reihen, Zinseszinsrechnung, Kombinatorik und Wahrscheinlichkeitsrechnung, Kettenbrüche, Stereometrie, Analytische Geometrie, Kegelschnitte, Maxima und Mimima. Leipzig, 1903. viii + 496 pp.

THIS work of 844 large pages is a most welcome addition to the few histories of elementary mathematics now easily acces-