# THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE eighteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, December 29 and 30, 1905. Morning and afternoon sessions were held on each day, the first session opening on Friday at 10:30 o'clock. The following members were present:

Professor D. F. Campbell, Dr. E. L. Dodd, Professor J. F. Downey, Professor J. W. Glover, Professor A. G. Hall, Professor Thomas F. Holgate, Dr. L. C. Karpinski, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Professor E. H. Moore, Professor H. B. Newson, Dr. J. C. Morehead, Miss Ida M. Schottenfels, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. J. Townsend, Mr. R. E. Wilson, Professor J. W. Young.

Professor A. G. Hall was elected chairman of the meeting. This being the regular time for the election of officers, the Section directed the nominating committee to report an executive committee for the Section to consist of a chairman, a secretary, and one additional member; this committee to have charge of all business of the Section including the preparation of the programme.

The officers elected for the year 1906 were as follows: Chairman, Professor Alexander Ziwet; Secretary, Professor H. E. Slaught; additional member of the executive committee, Professor A. G. Hall.

By vote the Section expressed its appreciation of the services of Professor Thomas F. Holgate, the retiring secretary, who had served the Section since its organization.

The executive committee, together with Professors Townsend and Skinner, were constituted a committee to arrange at an early meeting for a full half day conference of the Section with the deans and professors of engineering in technical schools, on the teaching of mathematics to engineering students.

The executive committee was requested to arrange so that the Section may take up the study of questions relating to applied mathematics, to the teaching and history of mathematics, and to the establishment of closer relations with colleagues in engineering and other professional subjects.

On the evening of Friday a dinner was held in the Hotel Del Prado, at which nineteen persons were present.

The following papers were read at this meeting:

(1) Dr. H. L. Coar: "Functions of three real variables."

(2) Dr. E. L. Dodd: "A theorem concerning implicit functions, and its geometric aspect."

(3) Professor Kurt Laves: "A dynamical interpretation of an integral of Jacobi's partial differential equation for the problem of a solid body rotating about a fixed point."

(4) Professor J. W. Young: "On discontinuous groups defined by the normal curve of the fourth order in a space of four dimensions."

(5) Professor E. H. Moore: "On the theory of systems of integral equations of the second kind" (preliminary communication).

(6) Professor J. W. Glover: "The teaching of actuarial theory in universities and colleges."

(7) Professor H. B. Newson: "Complex number systems and continuous groups" (preliminary communication).

(8) Dr. A. C. Lunn: "Sets of postulates for the trigonometric functions."

(9) Dr. J. C. Morehead: "Numbers of the forms $2^kq \pm 1$."

(10) Mr. N. J. Lennes: "Curves in non-metrical analysis situs."

(11) Mr. N. J. Lennes: "A fundamental existence theorem in the calculus of variations."

(12) Mr. N. J. Lennes: "Note on the Heine-Borel theorem."

(13) Dr. A. E. Young: "On certain isothermic surfaces."

(14) Professor J. B. Shaw: "Canonical forms of hypercomplex numbers" (preliminary communication).

(15) Professor Jacob Westlund: "Primitive roots of an ideal in an algebraic number field."

(16) Professor L. E. Dickson: "On quadratic, hermitian, and bilinear forms."

(17) Professor L. E. Dickson: "Expressions for the elements of a determinant in terms of the minors of a given order. Generalization of a theorem due to Studnička."

(18) Professor G. A. Miller: "The groups of order $p^m$ which contain exactly $p$ cyclic subgroups of order $p^\alpha$."

Dr. Coar was introduced by Professor Townsend. In the absence of the authors, the papers by Dr. Young, Professor Shaw, Professor Westlund, Professor Dickson, and Professor Miller were read by title.

Professor H. B. Newson introduced for general discussion the question of "The distribution of mathematical subjects during the first two years of the college course." This discussion was participated in generally by members of the Section and others present. The main question was the best possible arrangement of the work for these two years on the assumption that the student enters with algebra through the progressions and with plane and solid geometry. Professor Newson stated the problem before the state universities of the middle west, where in most places college and engineering students are taught in the same classes. The basis of the discussion was the new programme at the University of Kansas, and the opinion prevailed that college and engineering students should be given the same courses.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Coar spoke on functions of three real variables which are completely defined and single-valued for all values of the variables. In the case of such functions three kinds of limits exist: (1) the triple limit $\lim_{x \doteq x_0} \lim_{y \doteq y_0} \lim_{z \doteq z_0} f(x, y, z)$; (2) the double simultaneous limit $\lim_{x \doteq x_0} \lim_{y \doteq y_0, z \doteq z_0} f(x, y, z)$; (3) the simultaneous limit in three variables $\lim_{x \doteq x_0, y \doteq y_0, z \doteq z_0} f(x, y, z)$. Necessary and sufficient conditions for the existence of each of these were given. The conditions for the interchange of the order of sequence of these limits, of which there are six cases, were obtained, and questions of the continuity and discontinuity of the limiting functions were discussed. The function may converge uniformly to the limiting function in two ways, either with respect to one variable or with respect to two variables taken together. The existence of either of these cases involves that of the other. The adaptability of the ideas of limits to functions defined by a series of functions of two real variables was shown, as was the bearing of this on the problem of term by term integration in the case of simultaneous integrals.

2. Dr. Dodd pointed out that it is intuitionally evident that a continuous surface, passing through a point $(a, b)$ of the $(x, y)$-

plane, will cut the plane in a continuous line through $(a, b)$, provided the surface inclines in the same general direction near $(a, b)$. This suggests the following theorem regarding implicit functions, a theorem which may be proved analytically:

If $f(x, y)$ is a continuous function of $x$ and $y$, and a univariant function of $y$ near $(a, b)$, where $f$ vanishes, then there will exist one and only one continuous function $y = \phi(x)$ such that $\phi(a) = b$ and $f[x, \phi(x)] \equiv 0$ when $x$ is near $a$.

The hypothesis does not require that $\partial f/\partial y \neq 0$, nor even that this derivative should exist. The theorem can be generalized.

3. After Poisson had applied Lagrange's method of variation of constants to the problem in question, it was Richelot who, in 1850, obtained the same final differential equations of perturbed motion by applying to it the Hamilton-Jacobi method. The difficult task of finding a complete solution of Jacobi's partial differential equation necessitated very laborious transformations. By the introduction of a certain spherical triangle Radau brought about a simplification of the method. In Professor Laves's paper the attempt is made to accentuate the geometric aspects of the problem by giving to the integral under investigation a dynamical interpretation which seems to have escaped the attention of those who have worked on the problem.

4. Fricke has defined arithmetically an extensive class of discontinuous groups of linear fractional substitutions of the form

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

as the reproducing groups of ternary quadratic forms. Professor Young's paper aims at a generalization of Fricke's procedure. The conic represented by the ternary quadratic form is replaced by a normal curve $C_n$ of order $n$ in a space $S_n$ of $n$ dimensions. This curve is transformed into itself by a group of $\infty^3$ collineations in $S_n$, each collineation subjecting the points of $C_n$, which depend rationally on a parameter $\zeta$, to a substitution of the form $\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$. If the coefficients in the collineations are restricted to be integers with determinant unity, the collineation groups on the $z_i$, the point coördinates in $S_n$, and the holoedrically isomorphic group on the parameter $\zeta$ will be discontinuous. Thus every $C_n$ defines a discontinuous $\zeta$-group.

Professor Young first shows that the $\zeta$-groups defined as above by two normal $C_n$'s that can be transformed into each other by a rational collineation (*i. e.*, one with rational coefficients) are commensurable, *i. e.*, they have a subgroup of finite index in common; for this reason it is natural to confine oneself to the groups defined by $C_n$'s obtained from a canonical form by irrational collineations. The author then restricts himself to the case $n = 4$, and in particular to $C_4$'s obtained from the canonical form: $z_1 = \zeta^4$, $z_2 = \zeta^3$, $z_3 = \zeta^2$, $z_4 = \zeta$, $z_5 = 1$ by transformations of the form:

$$z_1' = \sqrt{p}\,z_1 + \sqrt{q}\,z_5, \qquad z_2' = \sqrt{m}\,z_2 + \sqrt{n}\,z_4, \qquad z_3' = \sqrt{r}\,z_3,$$
$$z_4' = \sqrt{m}\,z_2 - \sqrt{n}\,z_4, \qquad z_5' = \sqrt{p}\,z_1 - \sqrt{q}\,z_5,$$

where $p$, $q$, $m$, $n$, $r$ are positive integers. No essential restriction is involved in supposing that none of these integers contains a square factor and that $p$ is prime to each of the remaining four. It is then found that the $\zeta$-group $[p, q, m, n, r]$ will exist (*i. e.*, consist of more than the identical substitution) only if $r = p$, and further that this group is essentially the same as the reproducing group of the ternary quadratic form $Pz_1^2 - Qz_2^2 - Rz_3^2$, where $P$ is the highest common factor of $q$ and $m$, $Q$ the highest common factor of $m$ and $n$, and $R$ the highest common factor of $q$ and $n$.

5. Denoting (1) by $R$ a range of values of arguments or indices $s$, $t$; (2) by $F$ a field of ordinary complex numbers; (3) by $\Gamma_{1s}$ a class or category of single-valued functions $f_s$ of $s$ over $R$, with functional values belonging to $F$; (4) by $\Gamma_{2st}$ a category of single-valued functions $K_{st}$ of $(s, t)$ over $(R, R)$, with functional values belonging to $F$; (5) by $\widehat{t}$ an operation applicable to functions of $\Gamma_{1t}$ and serving to eliminate $t$ from such functions, — Professor Moore studies postulates involving the symbols $(R, F, \Gamma_1, \Gamma_2, \frown)$ in such a way as to yield a theory of the linear equation of the second kind,

$$g_s = f_s + \widehat{K_{st}^{\,t} f_t},$$

holding as $s$ ranges over $R$, a theory which suitably specialized becomes a theory of the system of linear integral equations of the second kind,

$$g_i(x) = f_i(x) + \sum_{j=1}^{m} \int_a^b K_{ij}(x, y) f_j(y) dy \qquad \left({i=1, \cdots, n \atop x=a, \cdots, b}\right).$$

8. Dr. Lunn's paper gave several sets of independent postulates which, with the accompanying definitions, sufficed to determine the sine, cosine, and tangent, as real functions of a real variable.

9. Dr. Morehead's paper is supplementary to his paper read at the April, 1905, meeting of the Society, on numbers of the forms $2^k q \pm 1$ and Fermat's numbers. For the purpose of determining the primes and the prime factors of non-primes of the forms $2^k q \pm 1$, two positive functions, $q_k(p)$, $q'_k(p)$, of the odd prime $p$ were defined. These functions possess the properties : $2^k q + 1 \equiv 0 \bmod p$, for $q = np + q_k$. $2^k q' - 1 \equiv 0 \bmod p$, for $q' = np + q'_k (n = 0, 1, 2, \cdots)$ and conversely. Thus for $q_{2^n} = 1$ we have $2^{2^n} \cdot 1 + 1 \equiv 0 \bmod p$, *i. e.*, $p$ is a factor of the Fermat number $F_n = 2^{2^n} + 1$. From the various properties of $q_k$, $q'_k$ is developed a means of computing $q_{2^n}$, which involves in general less labor than the usual method of computing successively the residues with respect to $p$ of $2^{2^1}$, $2^{2^2}$, $2^{2^3}$, $\cdots$ $2^{2^n}$ in order to determine whether or not $p$ is a factor of $F_n$.

The second part of the paper gives several theorems on the quadratic characters of $q_k$, $q'_k$ with respect to the corresponding prime $p$, and shows how some, and often all, of the quadratic residues and non-residues of $p$ can be written down at once.

10. In the first part of this paper Mr. Lennes gives fresh proofs of classical theorems on the separation of the plane by simple finite and infinite polygons and by a simply closed Jordan curve. The frontier of a region is considered and a set of conditions are specified under which such frontier is a Jordan curve. The definition of a continuous curve is at once translated into non-metrical language and the treatment is non-metrical throughout. In the second part of the paper several definitions of a continuous curve are given, one of which is developed as follows :

A set of points is connected if in every pair of complementary subsets at least one subset contains a limit point of points in the other set.

A set $\{P'''\}$ is said to separate a connected set $\{P\}$ into two sets $\{P'\}$ and $\{P''\}$ if every connected set lying in $\{P\}$ and containing points of both $\{P'\}$ and $\{P''\}$ also contains at least one point of $\{P'''\}$.

A closed, bounded, connected set of points containing the points $A$ and $B$ is a simple arc connecting $A$ and $B$ if every point of the set, except $A$ and $B$, separates it into two sets, one of which contains $A$ and the other $B$. Two simple arcs connecting $A$ and $B$ and having no point in common except $A$ and $B$ constitute a Jordan curve.

This definition of continuous curves applies equally well to curves in an $n$-dimensional manifold as to curves in the plane.

11. In this paper Mr. Lennes considers a set of arcs of Jordan curves connecting a fixed pair of points $A$ and $B$. The set of arcs lies in a closed region $R$, having a connected interior. The expression "entirely open region" is used to denote a set of points, no point of which is a limit point of points not in the set. An entirely open region of the region $R$ is a set of points of the region $R$ no part of which contains a limit point of points of $R$ which are not of the set.

In the region $R$ is considered the set $[C_{AB}]$ of all arcs connecting $A$ and $B$ which satisfies the following condition:

For every point $P$ of the region $R$ and for every triangle $t$ of which $P$ is an interior point there exists a triangle $t'$ of which $P$ is an interior point such that no arc of the set $[C_{AB}]$ contains a point on $t$ between two of its points on $t'$. (This may be regarded as a uniform continuity assumption on the set $[C_{AB}]$.)

Associated with each arc $C_{AB}$ of the set $[C_{AB}]$ is a positive number $M(C_{AB})$ forming a set of numbers $[M(C_{AB})]$ corresponding to the set of arcs $[C_{AB}]$. The set of numbers $[M(C_{AB})]$ is assumed to satisfy the following condition: For every given arc $C'_{AB}$ of the set $[C_{AB}]$ and for every given positive number $\epsilon$, there exists an entirely open region $R'$ of $R$, such that for every arc $C''_{AB}$ of $[C_{AB}]$ and of $R'$, $M(C'_{AB}) - M(C''_{AB}) < \epsilon$. With these assumptions it is proved that there exists at least one arc $C^{(m)}_{AB}$, of the set $[C_{AB}]$ such that $M(C^{(m)}_{AB})$ is the greatest lower bound of the set $[M(C_{AB})]$.

12. In this note Mr. Lennes formulates the Heine-Borel theorem on non-metrical hypotheses, as follows:

If in a three-dimensional continuous manifold every point of a closed set $[P]$ is an interior point (fails to be a limit point of points not of the set $[R]$) of at least one set of points $[R]$ of a set of sets $[[R]]$, then there exists a finite subset $[R]_1$,

$[R]_2, \cdots, [R]_n$ of the set $[[R]]$, such that every point of $[P]$ is an interior point of at least one of the sets $[R]_1, \cdots, [R]_n$.

The following corollary is a useful form for metrical analysis:

If every point of $[P]$ lies within at least one of the sets of $[[R]]$ then there exists a positive number $\delta$ such that every cube whose edge is $\delta$ and whose center is a point of the set $[P]$ lies within at least one of the sets of $[[R]]$.

13. In a paper presented at the sixteenth meeting of the Chicago Section (December, 1905), Dr. Young gave the result of a certain investigation leading to the consideration of some new isothermic surfaces, which, when referred to lines of curvature, have their linear element in the form $ds^2 = (u + v)^n(du^2/U + dv^2/V)$, where $n$ is a constant and $U$ and $V$ are functions of $u$ and $v$, respectively.

In the present paper he discusses the problem of determining all surfaces corresponding to the linear element above when referred to lines of curvature. Two distinct cases have to be considered, the one where $n = \pm 2$, and the other where $n$ has any other value. In the latter case Dr. Young shows first that the most general forms of the functions $U$ and $V$ are Laurent's series in $u$ and $v$, respectively. A further discussion of this case shows that if the functions $U$ and $V$ are assumed to be analytic, then $n = \pm 1$ and $U$ and $V$ take very simple forms. When $n = 1$, one form of the functions $U$ and $V$ gives the sphere, while another form gives the other quadric surfaces. Likewise, when $n = -1$, a certain minimal surface corresponds to the first form of the functions $U$ and $V$ and certain other surfaces peculiarly related to the other quadrics correspond to the second.

The surfaces corresponding to the case $n = -2$ are the cyclides of Dupin and certain other envelopes of spheres, whereas the surfaces corresponding to $n = +2$ are the new surfaces which were considered in the previous paper.

This discussion is of particular interest in that it gives a good example of the comparative ease with which positive results may be obtained by working from the Gauss and Codazzi equations. All the quadratures are performed and the cartesian coördinates of all the new surfaces are given as functions of $u$ and $v$.

14. Professor Shaw's paper develops the theorem that any

number in any linear associative algebra can be written in the form

$$\phi = \sum_{i,j} A_{i,j}^{(s \cdots \mu_i)} \eta_{i,j} \qquad (i, j = 1, \cdots, \rho),$$

where $\eta_{i,j}$ are quadrate vids, subject, that is, to the law $\eta_{ij}\eta_{jk} = \eta_{ik}$; $\eta_{ij}\eta_{kl} = 0$ if $j \neq k$. The quantities $A_{i,j}^{(s \cdots \mu_i)}$ are polynomials in a nilpotent $\nu$, such that the power $\nu^{\mu_1+1} = 0$; the coefficients of the powers of $\nu$ are from any field, scalar, abstract, etc. The powers of $\nu$ in $A^{(s \cdots \mu_i)}$ start with the $s$th and run consecutively to the $\mu_i$th. The number $s$ is such that $s = 0$ if $\mu_i \leqq \mu_j$, $s = \mu_i - \mu_j + 1$ if $\mu_i > \mu_j$. The numbers $\mu_i$ $(i = 1, \cdots, \rho)$ are integers assigned beforehand and are such that $\mu_1 \geqq \mu_2 \geqq \mu_3 \cdots \geqq \mu_\rho$.

The consequences of this theorem are developed partially; in particular, for Peirce algebras, that is, non-quaternion algebras with no skew units. Such algebras are generated by $\rho$ linearly independent, but not necessarily productly independent units. When their polynomials are known the algebra is determined. The persistent coefficients of these polynomials under linear transformation determine the different algebras of the same type, in terms of their "natural units." This paper is a continuation of "Theory of linear associative algebra" and "Nilpotent algebras" by the author (*Transactions*, Volume 4, 1903).

15. Professor Westlund's paper is a continuation and extension of a former paper. It is shown that in an arbitrary algebraic number field the only ideals which have primitive roots are of the form $P^n$, $Q_1^{a_1} Q_2 \cdots Q_i$, and $Q_1 Q_2 \cdots Q_i P^n$, where $p$, the rational prime divisible by $P$, is odd, and $Q_1, Q_2, \cdots$ are the prime ideal factors of 2. A method is also given for determining the primitive roots in each case.

16. The first paper by Professor Dickson treats of the reduction to canonical types of quadratic and hermitian forms in a general field. Application is made to quadratic and bilinear forms invariant under a given substitution $S$ and their normalization by substitutions commutative with $S$. The latter problem is a generalization to an arbitrary finite or infinite field of the recent memoir by Jordan in the *Journal de Mathématiques* on the case of a finite field of prime order. The paper will appear in the *Transactions*, April, 1906.

17. The second paper by Professor Dickson gives a practical method of expressing the elements of a determinant $D$ of order $n$ as rational functions of the minors $M_m$ of order $m$ and the $m$th root of a rational function of the $M_m$. If from $m-1$ rows of $D$ we build $n-m$ minors $M^{(i)}_{m-1}$ such that the elements of $M'$ are chosen from any columns other than a fixed column $C_{j_1}$, the elements of $M''$ from any columns other than $C_{j_1}$ and $C_{j_2}$, etc., we can express $M'M'' \cdots M^{(n-m)}D$ as a determinant of order $n-m+1$ whose elements are minors $M_m$ of $D$. For the case $M' = M'' = \cdots = M^{(n-m)}$ the theorem is due to Studnicka. The latter case enables us to derive the minors $M_{m-1}$ from the $M_m$, then the $M_{m-2}$ from the $M_{m-1}$, and finally the elements from the $M_2$. This symmetric solution is valueless since it employs a complicated array of radicals of various degrees. The other solutions given involve a single irrationality, viz., the $m$th root of a rational function of the $M_m$. The paper appeared in the *American Mathematical Monthly*, December, 1905.

18. Professor Miller considers the possible types of groups $G$ of order $p^m$ ($p$ any prime) which contain just $p$ cyclic subgroups of a given order $p^\alpha$ ($\alpha < m$). He finds that every group $G$ of this kind must contain a cyclic subgroup of order $p^{m-2}$. If $p$ is restricted to be odd and $m > 4$, a stronger theorem is obtained as follows: When $p$ is odd and $m > 4$, $G$ contains exactly $p$ cyclic subgroups of every order from $p^2$ to $p^{m-1}$. This theorem is also true when $m = 3$, and when $m = 4$ and $p > 3$. As all the groups of order $p^m$ which contain a cyclic subgroup of order $p^{m-2}$ are known, these results give a complete determination of all the groups of order $p^m$, which contain exactly $p$ cyclic subgroups of the same order.

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*Secretary of the Section.*

EVANSTON, ILLINOIS,
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